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ABSTRACT

Many mathematics educators and researchers agree that current mathematics instruction in elementary and secondary schools focuses too much on efficient computation and not enough on mathematical understanding, problem solving, and reasoning. But beyond this agreement exist diverse views about what it means to know, understand, and learn mathematics. This paper considers various research perspectives that can inform thinking about what mathematics should be like in U.S. public elementary schools. To this end, the paper examines recent National Council of Teachers of Mathematics recommendations for changes in the mathematics curriculum and three major research perspectives that may inform thinking about what it means to know, understand, and learn mathematics. The first of these perspectives is that of researchers, primarily cognitive psychologists, who focus on the individual knower of mathematics. The second perspective is that of the discipline of mathematics, what it means to understand mathematics from within the discipline. The third perspective is that of "classroom practice," in particular the ways in which researchers studying the teaching and learning of mathematics in classrooms have conceptualized mathematical knowledge and learning. Goals for mathematics learning are discussed. A list of 108 references is included. (Author/YP)

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Elementary Subjects Center
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ALTERNATIVE PERSPECTIVES ON KNOWING MATHEMATICS IN ELEMENTARY SCHOOLS

Ralph T. Putnam, Magdalene Lampert,
and Penelope L. Peterson

Center for the Learning and Teaching of Elementary Subjects

Institute for
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Center for the Learning and Teaching of Elementary Subjects

The **Center for the Learning and Teaching of Elementary Subjects** was awarded to Michigan State University in 1987 after a nationwide competition. Funded by the Office of Educational Research and Improvement, U.S. Department of Education, the Elementary Subjects Center is a major project housed in the Institute for Research on Teaching (IRT). The program focuses on conceptual understanding, higher order thinking, and problem solving in elementary school teaching of mathematics, science, social studies, literature, and the arts. Center researchers are identifying exemplary curriculum, instruction, and evaluation practices in the teaching of these school subjects; studying these practices to build new hypotheses about how the effectiveness of elementary schools can be improved; testing these hypotheses through school-based research; and making specific recommendations for the improvement of school policies, instructional materials, assessment procedures, and teaching practices. Research questions include, What content should be taught when teaching for conceptual understanding and higher level learning? How do teachers concentrate their teaching to use their limited resources best? and In what ways is good teaching subject matter-specific?

The work is designed to unfold in three phases, beginning with literature review and interview studies designed to elicit and synthesize the points of view of various stakeholders (representatives of the underlying academic disciplines, intellectual leaders and organizations concerned with curriculum and instruction in school subjects, classroom teachers, state- and district-level policymakers) concerning ideal curriculum, instruction, and evaluation practices in these five content areas at the elementary level. Phase II involves interview and observation methods designed to describe current practice, and in particular, best practice as observed in the classrooms of teachers believed to be outstanding. Phase II also involves analysis of curricula (both widely used curriculum series and distinctive curricula developed with special emphasis on conceptual understanding and higher order applications), as another approach to gathering information about current practices. In Phase III, test models of ideal practice will be developed based on what has been learned and synthesized from the first two phases.

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Abstract

From all sides are coming calls for changes in the amount and quality of mathematics instruction in American schools. Mathematics educators and researchers agree that current mathematics instruction in elementary and secondary schools focuses too much on efficient computation and not enough on mathematical understanding, problem solving, and reasoning. But beyond this agreement exist diverse views about what it means to know, understand, and learn mathematics. In this paper, the authors consider various research perspectives that can inform thinking about what mathematics should be like in our public elementary schools. To this end, they examine recent recommendations for changes in the mathematics curriculum and three major research perspectives that may inform thinking about what it means to know, understand, and learn mathematics. The first of these perspectives is that of researchers, primarily cognitive psychologists, who focus on the individual knower of mathematics. The second perspective is that of the discipline of mathematics--what it means to undertand mathematics from within the discipline. The third perspective is that of classroom practice, in particular the ways in which researchers studying the teaching and learning of mathematics in classrooms have conceptualized mathematical knowledge and learning.

ALTERNATIVE PERSPECTIVES ON KNOWING MATHEMATICS IN ELEMENTARY SCHOOLS

Ralph T. Putnam, Magdalene Lampert, and Penelope L. Peterson[1]

This is one of a set of seven reports being prepared for Study 1 of Phase I of the research agenda of the Center for the Learning and Teaching of Elementary Subjects. Phase I of our work calls for surveying and synthesizing the opinions of various categories of experts concerning the nature of elementary-level instruction in mathematics, science, social studies, literature, and the arts, with particular attention to how teaching for understanding and for higher order thinking and problem solving should be handled within such instruction. Study 1 of Phase I calls for review of the literature in educational psychology, cognitive science, and related fields on teaching for understanding and for higher order thinking and problem solving, as well as the literature on these topics as they are discussed by curriculum and instruction experts within the context of teaching particular school subjects. The present paper focuses on various research perspectives that can inform our thinking about what it means to know and understand mathematics in powerful ways at the elementary-school level.

From all sides are coming calls for changes in the amount and quality of mathematics instruction in American schools (National Commission on Excellence in Education, 1983; National Council of Teachers of Mathematics, 1980; National Research Council, 1989). Critics of current practice posit that the mathematical achievement and understanding of United States students lag behind that of their peers in other industrialized countries (McKnight et al., 1987; National Commission on Excellence in Education, 1983; National Science Board Commission on Pre-College Education in Mathematics Science and Technology, 1983). Mathematics educators and researchers argue that current mathematics

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instruction in elementary and secondary schools focuses too much on efficient computation and not enough on mathematical understanding, problem solving, and reasoning. Leaders in business and industry are claiming that public education must change to teach to the new kinds of mathematical skills and problem solving abilities that will be important for the worker of the future (see, e.g., Bernstein, 1988). Accompanying these criticisms of current practice are calls for reform--for making lasting and fundamental changes in mathematics curriculum and instruction in public schools. New guidelines and standards for mathematics curriculum, instruction, and assessment have been proposed by the National Council of Teachers of Mathematics (1989). Some states, most notably California (California State Department of Education, 1987), have developed or are developing new guidelines aimed at the reform of mathematics curriculum and educational practice.

Two sets of beliefs about why the learning of mathematics is important motivate current reform efforts. These beliefs parallel traditional but differing goals of schooling. First is the belief that mathematics provides essential tools and ways of thinking in our society, including those needed for a successful labor force and for an informed citizenry. Second is the belief that mathematics is important for self-fulfillment and appreciation of one of humanity's great cultural achievements.

One major argument for the reform of mathematics curriculum and instruction is that with advances in technology and information systems, the needs of the labor force in our society have changed, but the learning and teaching of mathematics in our nation's schools have not shifted correspondingly to meet these needs. To be competitive in the global economy, our country needs a well prepared, productive work force. And our current instructional system is not providing students with the kind of mathematics they need to be productive. The ready availability of calculators and computers, along with their increasing role in the workplace, has raised serious questions about what kinds of mathematical knowledge and skills should be considered basic. In the past, an important goal of elementary school was to help a large number of students become proficient in arithmetic calculation; these skills were considered basic for a skilled labor force whose tasks were largely carrying out repetitive and routine tasks. But with most of the repetitive production-oriented tasks being taken over by automation and by cheap labor in less developed countries, the demands on the American work force have shifted dramatically. To be productive in the

workplace and to be informed citizens in our information-oriented society requires more sophisticated mathematical skills and knowledge, particularly the ability to communicate with mathematical systems and to solve a variety of complex problems.

The second argument is essentially that mathematics should be learned and used, not for any direct utilitarian purpose, but because it is a great achievement of human thinking that should be appreciated and shared with others. Knowing mathematics is an important part of what it means to be an educated person. This set of arguments is less visible than the more utilitarian ones in the rhetoric of the current reform, but it is important for at least two reasons. First, much of what we value in the elementary school curriculum is justified, at least in part, by appeal to self-fulfillment and appreciation of cultural achievement. The obvious examples are history, literature, art, and music, but all subjects are valued, to some extent, because of their cultural significance. The second reason is that gaining a broad appreciation for mathematics and learning the more general ways of reasoning within it are essential to the powerful and flexible mathematical thinking and problem solving demanded by today's information-oriented society.

Resnick and Resnick (1977, see also, Resnick, 1987a) make a similar argument by pointing out the traditional tension between two traditions of schooling and literacy: A low-literacy tradition for training the masses to produce a competent work force has existed in parallel with a high-literacy tradition for an intellectual elite. Resnick and Resnick argue that the powerful and flexible thinking and reasoning that have always been the goal of education in the elite educational systems have become the basic educational goals for the masses. The challenge today is to come up with ways to work toward these difficult goals, traditionally reserved for the elite, with all students in public schools.

In this paper we consider various research perspectives that can help us think about these and other arguments for what mathematics instruction should be like in our public elementary schools. To this end, we first examine one important set of recommendations for school mathematics, the Curriculum and Evaluation Standards for School Mathematics, recently proposed by the National Council of Teachers of Mathematics [NCTM] (1989). At the heart of this document and most other calls for reform (National Council of Teachers of Mathematics, 1980; National Research Council, 1989) is the view that current elementary school mathematics curriculum overemphasizes efficient computational arithmetic skill at the

expense of understanding and problem solving. Most researchers and mathematics educators agree that there is more to mathematics than computational proficiency.

But beyond this agreement exist diverse views about what it means to know, understand, and learn mathematics. These views emerge from various research traditions with differing goals, assumptions, and questions. After discussing the NCTM curriculum standards, we consider three major perspectives that may inform our efforts to understand what it means to know, understand, and learn mathematics. First, we consider the views of researchers and scholars, primarily psychologists, who focus on what it means to know and understand mathematics from the perspective of the individual knower. Then we consider perspectives on what it might to understand mathematics from the within the discipline of mathematics. Finally, we consider views on knowing mathematics from the perspective of classroom practice. How have researchers studying the teaching and learning of mathematics in classrooms conceptualized mathematical knowledge and what views have emerged from those engaged in teaching mathematics to others?

A Statement of Goals: NCTM Curriculum Standards

The Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989) represents the latest in a series of statements by the mathematics education community about what mathematics should be taught in public schools. The document is NCTM's response to the numerous calls for reform in mathematics education (National Commission on Excellence in Education, 1983; National Science Board Commission on Pre-College Education in Mathematics Science and Technology, 1983; Romberg, 1984). The Standards document offers a vision of school mathematics that is consistent with earlier statements by NCTM and other professional organizations about ideal curriculum and instruction in mathematics (National Council of Supervisors of Mathematics, 1977; National Council of Teachers of Mathematics, 1980). However, NCTM has gone beyond these organizations to actually set standards of reform. Unlike countries in which a national curriculum exists in mathematics (e.g., Japan, United Kingdom, China), the United States has no such national curriculum or even national "vision." The NCTM Standards represent an attempt to develop such a vision at the national

level. The recommendations are also consistent with the current efforts of the National Research Council's Mathematics Sciences Education Board (National Research Council, 1989) to rethink school mathematics from the ground up.

The NCTM (1989) Standards call for major changes in (a) the content of school mathematics and (b) the nature of mathematics instruction and underlying view of mathematics learning. According to NCTM, the elementary mathematics curriculum should be broadened beyond its traditional focus on arithmetic computation to include more emphasis on conceptual understanding, and on currently underrepresented mathematical domains such as geometry, measurement, and statistics. The justifications offered for these changes are largely utilitarian, focusing on the need for transformation in the kinds of mathematics that students will need in a technological, information-oriented society. The Standards authors argue that shifting from an industrial-based to an information-based society "has transformed both the aspects of mathematics that need to be transmitted to students and the concepts and procedures they must master if they are to be self-fulfilled, productive citizens in the next century" (p. 3).

Technology, in the form of computers and calculators, has fostered important changes in the content of mathematics and its applications to other disciplines, as well as causing radical changes in the workplace. Because of the growing use of computers, with their ability to manipulate huge amounts of information, quantitative approaches and techniques have become increasingly important in many disciplines. The mathematical concepts and models underlying many of these approaches, however, are not necessarily those that are emphasized in the traditional school curriculum. In addition, technology has resulted in important changes within the discipline of mathematics itself, through the facilitation of calculations and graphing, and by changing the nature of problems that mathematicians address. The traditional school curriculum has not been modified to reflect these important changes.

The nature of classroom instruction should move away from the traditionally prevalent model of teacher as teller and students as passive recipients of mathematical knowledge to an emphasis on learning mathematics through problem solving, discussion, and other practices consistent with the notion that students need to be actively involved. As justification, the Standards authors offer some changing features of mathematics and views on mathematical knowledge. They posit a changing view about the

nature of mathematical knowledge; in particular, the view that " 'knowing' mathematics is 'doing' mathematics" (NCTM, 1989, p. 7). Rather than viewing mathematics learning as the mastery of concepts and procedures, the Standards authors assert that such "informational knowledge" has value only to the "the extent to which it is useful in the course of some purposeful activity" (p. 7). Thus, instruction should always emphasize the acquisition of and use of knowledge in the context of purposeful activity, such as problem solving, in contrast to the traditional view of mathematics teaching in which computational facts and algorithms are learned first as prerequisite skills to be applied later in the solving of problems.

In this context, NCTM (1989) offers four general social goals for education in the area of mathematics:

1. Mathematically literate workers. The technologically demanding workplace of today and the future will require mathematical understanding and the ability to formulate and solve complex problems, often with others. "Businesses no longer seek workers with strong backs, clever hands, and 'shopkeeper' arithmetic skills" (p. 3).
2. Lifelong learning. Most workers will change jobs frequently, and so need flexibility and problem-solving ability to enable them to "explore, create, accommodate to changed conditions, and actively create new knowledge over the course of their lives" (p. 4).
3. Opportunity for all. Because mathematics has become "a critical filter for employment and full participation in our society" (p. 4), it must be made accessible to all students, not just the white males, the group that currently studies the most advanced mathematics.
4. An informed electorate. Because of the increasingly technical and complex nature of current issues, participation by citizens requires technical knowledge and understanding, especially skills in reading and interpreting complex information.

These social goals require that students become mathematically literate, a key phrase used by the Standards authors to describe desired outcomes of schooling. Mathematical literacy "denotes an individual's abilities to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" (NCTM, 1989, p. 5). The authors of Everybody Counts (National Research Council, 1989) argue that "without the ability to understand basic mathematical ideas, one cannot fully comprehend modern writing such as that which appears in the daily

newspapers" (p. 7). They go on to emphasize that mathematical literacy includes much more than familiarity with numbers and arithmetic: "To cope confidently with the demands of today's society, one must be able to grasp the implications of many mathematical concepts--for example, chance, logic, and graphs--that permeate daily news and routine decisions" (pp. 7-8). This view contrasts sharply with the implicit traditional view of computational skills in arithmetic as forming the core of the basic skills needed to function effectively in the workplace and society.

The Standards (NCTM, 1989) authors articulate the notion of mathematical literacy by proposing five general goals for students:

1. Learning to value mathematics--understanding its evolution and its role in society and the sciences.
2. Becoming confident of one's own ability--coming to trust one's own mathematical thinking and having the ability to make sense of situations and solve problems.
3. Becoming a mathematical problem solver--which is essential to becoming a productive citizen and which requires experience in solving a variety of extended and nonroutine problems.
4. Learning to communicate mathematically--learning the signs, symbols, and terms of mathematics.
5. Learning to reason mathematically--making conjectures, gathering evidence, and building mathematical arguments.

These goals reflect a shift away from the traditional practice of summarizing desired mathematical outcomes as knowledge of skills, concepts, and applications (Fey, 1982; Trafton, 1980) to an emphasis on broader dispositions, attitudes, and beliefs about the nature of mathematical knowledge and about one's own mathematical thinking. The traditional skills, concepts, and applications are subsumed under the more general goals for problem solving and communication. Throughout the Standards document, the authors de-emphasize the view that knowledge consists of distinct parts that should be treated separately. Rather, they emphasize providing students with experiences through which they can build rich connections among the various kinds of knowledge.

The curriculum standards themselves are presented as sets of standards for elementary school (Grades K-4), middle-school (Grades 5-8), and high school (Grades 9-12). Each set begins with an common set of four standards reflecting the five general goals just described: (a) mathematics as problem solving, (b) mathematics as communication, (c) mathematics as reasoning, and (d) mathematical connections. The inclusion of a connections standard focuses attention on the authors' "belief that although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole. Concepts, procedures and intellectual processes are interrelated" (NCTM, 1989, p. 11).

The four general standards are followed by more specific content standards for each cluster of grade levels. For elementary school these content standards address estimation, number sense and numeration, concepts of whole number operations, whole number computation, geometry and spatial sense, measurement, statistics and probability, fractions and decimals, and patterns and relationships. The middle school standards address number and number relationships; number systems and number theory; computation and estimation; patterns, relations, and functions; algebra; statistics; probability; geometry; and measurement.

In terms of mathematical content, these standards reflect major shifts from the content and emphasis of current mathematics curricula. Most significant is a reduced emphasis on arithmetic computation, especially mastery of complex paper-and-pencil algorithms, with a shift in focus to meaning and appropriate use of operations, judging the reasonableness of results, and choice of appropriate procedures. Along with this shift is an emphasis on problem solving, including use of word problems with a variety of structures, everyday problems, strategies for solving problems, and open ended problems that take more than a few minutes to solve. Mathematical topics that are considered increasingly important--but seriously underrepresented in current curricula--include geometry and measurement, probability and statistics; and patterns, relationships, and functions. For later elementary grades, algebra is included, with less focus on manipulation of symbols and memorization of rules and more focus on informal investigation and understanding of variables, expressions, and equations.

The Standards document presents a vision of what mathematics instruction might be like. Although many of its recommendations could be supported by existing or future research, the Standards is not a research document. Many of the recommendations in it express shared assumptions and views of many members of the mathematics education community that have not been (or in some cases could not be) addressed by research. In the remainder of this paper, we consider research and scholarship that can inform our thinking about recommendations like those in the NCTM Standards, organizing our discussion around groups of scholars focusing on the individual knower, on the discipline of mathematics, and on the classroom context. In considering these perspectives, it is not our intent to provide clear answers to questions about what mathematics should be taught in elementary schools and how that learning should proceed. Rather, we hope to raise questions and issues to inform debate about these issues and to avoid ignoring important aspects of knowing and learning mathematics that might result from arguing for change from a single perspective.

Focus on the Individual Knower: Cognitive Psychology

In considering the individual learner and knower of mathematics, we focus on cognitive perspectives that have come to dominate mainstream American psychology. The research and views we consider here include both those of psychologists using mathematics as a site for inquiry about basic issues in learning and those of researchers primarily interested in the learning of mathematics who draw on the perspectives and tools of cognitive psychology.

The dramatic shift in mainstream American psychology from its associationist and behaviorist traditions to the study of cognition has important implications for thinking about learning and teaching in schools (Calfee, 1981; Resnick, 1985; Shuell, 1986). Whereas behaviorist psychologists insisted that observable behavior was the only legitimate object of scientific study, cognitive psychologists treat "thinking processes as concrete phenomena that can be studied scientifically" (Resnick, 1985, p. 128). They continue to ground their work in observing the behavior of individuals, but use these observations as evidence for positing various cognitive structures and processes believed to produce the behavior. In thinking about learning, cognitive theorists consider learning to be changes in the knowledge or cognitive

processes that produce behavior, in contrast to the behaviorist position that the learning is a change in the behavior itself (Shuell, 1986). Associationists (e.g., Thorndike, 1922) were willing to hypothesize cognitive events but built their theories of learning and knowledge around the notion of stimulus-response bonds as the building blocks of knowledge. This reductionist approach produced a view of knowledge as collections of bonds, which were combined to produce more complex forms of knowledge. Current cognitive theorists, in contrast, place much more emphasis on the structure of and relationships among various kinds of knowledge--not on knowledge as collections of discrete bits.

One important result of this shift in focus has been increased attention to and tools for studying difficult issues in learning and knowing, such as the nature of understanding and complex forms of knowledge. Whereas much psychological research on learning in the past focused on studies of isolated learning tasks such as nonsense syllables, many cognitive psychologists have turned to school-relevant domains such as reading and mathematics as the domains of inquiry about learning. Mathematics has served as an important site for much of this research for a variety of reasons: Mathematics' foundational role in many other disciplines makes it a prime target for understanding basic processes in thinking and knowing; much of mathematical knowledge lends itself to specification in the precise forms needed for the computational models that form the basis of cognitive science; and mathematics (or at least arithmetic) is considered an important basic skill in the school curriculum.

An important result of this shift in focus toward cognitive processes and complex forms of knowledge has been the development of methods to examine and describe patterns of thought and knowledge, often at fine levels of detail. Methods that cognitive psychologists have used to examine what individuals do and think about as they carry out various tasks include recording reaction times or eye movements or having individuals "think aloud." Cognitive researchers use these data as a basis for hypothesizing in detail the knowledge and thought processes believed to underlie individuals' performance.

The computer has served both as metaphor and tool in the building of these theories of the knowledge and processes hypothesized to underlie performance. A basic assumption for many cognitive psychologists is that the human mind, like the computer, is essentially a processor of information. The mind receives information from the environment through the senses and processes and transforms that

information. This function is similar to that performed by computers, which also process information through complex structures. The power of the computer metaphor for human thought is in its leading to precise hypotheses about how information is represented and processed in the mind. It is in building these precise hypotheses that the computer also serves as an important research tool for cognitive scientists. Some cognitive researchers write computer programs that are fine-grained simulations of human thinking. The writing of these programs promotes a certain rigor in the description of cognition.

> When computer programs behave as humans do--making similar mistakes, pausing at similar points, expressing confusion over the same issues--it is reasonable to assume that the internal processes of the human and the computer are similar, and researchers can treat the program's visible processes as a theory of the invisible processes of humans. (Resnick, 1985, p. 129).

Trying to characterize human thought in the form of computer programs has given rise to a variety of constructs for representing and describing hypothesized knowledge in people's minds (e.g., production systems, semantic networks, and schemata).

Even when they do not specify hypothesized knowledge structures in the form of computer programs, virtually all cognitive theorists share the fundamental assumption that an individual's knowledge structures and mental representations of the world play a central role in perceiving, comprehending, and acting (Shuell, 1986). An individual's perception of the environment and his or her actions are mediated through his or her cognitive structures, which are actively constructed and modified through the individual's interaction with the environment. This mediation through cognitive structures provides a basic, though overly simplified, definition of knowledge in cognitive theories: Knowledge <u>is</u> the cognitive structures of the individual knower. To know and understand mathematics from this perspective means having acquired or constructed appropriate knowledge structures.

But the story is more complicated than that. From this basic view of knowledge there have emerged a host of more specific views of what it means to know and understand mathematics. We will structure our discussion of these views around five themes. Because the themes are so highly interrelated, the order in which we discuss them is somewhat arbitrary; it will be impossible to discuss any one of the themes without bringing in aspects of the other three.

The first theme is understanding as representation, in particular the view that understanding mathematics means having internalized powerful symbols and systems for representing mathematical ideas and being able to move fluently within and between them. Because issues of representation are so fundamental to cognitive psychology and mathematics, we will also discuss representation in more general terms, foreshadowing several issues addressed later in the paper.

The second theme is understanding as knowledge structures. A large portion of research in cognitive science has been directed at describing the knowledge, in the form of cognitive structures and processes, hypothesized to underlie competent performance on various mathematical tasks. This approach builds directly on the basic view of understanding as an individual having constructed or acquired appropriate knowledge structures.

In discussing the knowledge structure of individuals, some researchers have emphasized a third theme, understanding as connections among types of knowledge. Of particular interest are connections between conceptual and procedural knowledge and between knowledge of the formal, symbolic mathematics taught in school and the rich base of informal knowledge children develop in out-of-school settings.

Researchers emphasizing the fourth theme, learning as the active construction of knowledge, have highlighted the nature of the process by which knowledge structures have been constructed or acquired by individuals. Learning mathematics with understanding from this perspective means actively reorganizing one's cognitive structures and integrating new information with existing structures.

The fifth theme, understanding as situated cognition, represents a growing movement within cognitive science to question the fundamental view of thinking and knowing underlying current cognitive theories. Rather than viewing knowledge and thinking as existing within the mind of the individual, cognition is considered to be interactively situated in physical and social contexts.

Understanding as Representation

Because the notion of representation is fundamental to both cognitive psychology and mathematics, various forms of representation are central to cognitive research on mathematics knowing and learning.

One could say that cognitive psychology is about hypothesizing the sorts of mental representations that individuals have and use. At the same time, "the idea of representation is continuous with mathematics itself" (Kaput, 1987a, p. 25). Virtually all of mathematics concerns the representation of ideas, structures, or information in ways that permit powerful problem solving and manipulation of information. Thus, when one is considering the nature of knowing and learning mathematics from the perspective of cognitive psychology, issues of representation are unavoidable. In part because of its pervasiveness in this work and its role in fundamental assumptions, representation is a slippery term, like the term concept, that is used in a variety of related ways and defies precise definition (Kaput, 1985).

Kaput (1985, 1987a, 1987b), in considering the various roles of representation in learning, knowing, and doing mathematics, as well as the role of representation in posing psychological models of these phenomena, has suggested as important the following broad interacting types of representation: (a) cognitive representation, the representation of information or knowledge in the mind of the individual; (b) explanatory representation, the models that psychologists pose to describe hypothesized mental structures and events; (c) mathematical representation, the representation of one mathematical structure by another; and (d) external symbolic representation, the material forms used to express abstract mathematical ideas. As defined by Kaput (1985), following Palmer (1977), each of these forms of representation involves a represented world (the thing being represented), a representing world (the thing doing the representing), and correspondences between selected aspects of these two worlds.

Cognitive and explanatory representation are at the heart of cognitive psychology. The goal of cognitive psychologists is to construct models (explanatory representations) of what they hypothesize to be the ways information or knowledge is stored and acted upon within the minds of individuals (cognitive representations). Although psychologists often fall into using language which suggests that their models of cognition are descriptions of structures that actually exist inside the head of individuals, it is important to keep in mind that all psychologists' models are theories about cognitive structures and events that are, in principle, unobservable. In addition, the basic assumption in most cognitive psychology that individuals build cognitive representations of an external world runs into an epistemological dilemma: Because the external world can be known only through these same cognitive representations, there is no way of

knowing what is "really" out there to be represented. In other words, it is impossible to assess the match between the representing world and the represented world (see Kaput, 1987b; von Glasersfeld, 1987).

Most researchers taking a cognitive approach in studying the knowing and learning of mathematics, however, ignore this basic philosophical dilemma by making the working assumption that it is useful to analyze and build models (representations) of the information structures and processes underlying the knowing and doing of mathematics. As we argued earlier, the (often implicit) view of what it means for an individual to know mathematics from this perspective is for that individual to have acquired or constructed the appropriate cognitive representations. We consider cognitive researchers' attempts to characterize these knowledge structures in our subsequent discussion of the themes, understanding as knowledge structures and understanding as connections among types of knowledge.

Regarding mathematical representations, Kaput (1987a) argues that much of mathematics involves the representation of one mathematical structure by another and determining what is preserved and what is lost in the mapping between the structures. An example at a level relevant for thinking about elementary schooling is that much of algebra can be seen as representing in a general way many more particular arithmetic relationships. Kaput argues that mathematical structures are treated as abstractions or idealizations that are formally independent of the material symbols used to represent them. For example, there is a sense in which the number 3 is assumed to exist, independent of whether the number is represented by the word three, by the numeral 3 or by three dots, for example, • • •.

But because mathematical entities and structures are abstract, they must be expressed in some material form, and that is the role played by external symbolic representations. We need symbolic representations both to support our personal thinking about mathematical ideas and to communicate with others about them (Kaput, 1987b). The representations that can be used to support thinking and communicating about mathematics include not only the formal symbol systems of mathematics, such as the base-10 notational system and the Cartesian coordinate system, but more informal systems of representation as well. For example, Lesh, Post, and Behr (1987) consider the following kinds of representation systems as being important: (a) experienced-based scripts, (b) manipulable models, (c) pictures or diagrams, (d) spoken languages, and (e) written symbols. All of these representation

systems can be thought of as powerful tools that are developed as part of a culture and that become incorporated into the cognitive systems of individuals (Stigler & Baranes, 1988). Representation systems thus play roles similar to those of natural language in supporting personal thought and public communication. Like natural language, these symbol systems have a dual existence: There is a sense in which they exist as personal constructions in the mind of the individual and a sense in which they exist external to the individual as a product of the discourse or cultural community. One way of thinking about the goals of schooling in mathematics is for the learner to construct or internalize the shared symbol systems of mathematics.

Some researchers have emphasized the importance of being able to move flexibly both within particular representation systems and to make translations across representation systems (Janvier, 1987). Lesh et al. (1987) argued that

> Part of what we mean when we say that a student "understands" an idea like "1/3" is that (1) he or she can recognize the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within given representational systems, and (3) he or she can accurately translate the idea from one system to another. (p. 36)

Lesh et al. offered the following example of a student moving fluently back and forth among various representations while solving a problem:

> The Million Dollar Problem: Imagine that you are watching: "The A Team" on television. In the first scene, you see a crook running out of a bank carrying a bag over his shoulder, and you are told that he has stolen one million dollars in small bills. Could this really have been the case?
>
> One student who solved this problem began by using sheets of typewriter paper to represent several dollar bills. Then, he used a box of typewriter paper to find how many $1 bills such a box would hold--thinking about how large (i.e., volume) a box would be needed to hold one million $1 bills. Next, however, holding the box of typewriter paper reminded him to think about weight rather than volume. So, he switched his representation from using a box of typewriter paper to using a book of about the same weight. By lifting a stack of books, he soon concluded that, if each bill was worth no more than $10, then such a bag would be far too large and heavy for a single person to carry. (p. 39)

The basic argument here is that learners acquire as personal cognitive tools the powerful ways of representing mathematical ideas that are used in our culture. Because of this and because teaching and learning inherently require communication, the ways in which mathematical ideas are represented externally in the classroom become crucial. Many researchers use the term instructional representations for the ways that teachers or curriculum materials represent mathematical knowledge (e.g., Ball, 1988;

Greeno, 1987a; Wilson, 1988). These and other researchers have begun to consider a host of important pedagogical issues in the use of instructional representations, including accessibility of particular instructional representations to individual students, the motivational characteristics of representations, and correspondences (and mismatches) between instructional representations and the mathematics they are intended to represent. Because these important issues deal directly with pedagogy, rather than the nature of mathematical understanding, we do not deal with them directly in this paper.

Understanding as Knowledge Structures

We have argued that a basic view of what it means to know from the perspective of cognitive psychology is for an individual to have acquired appropriate knowledge structures. From this perspective, thinking and knowing take place within the mind of the individual; interaction with the environment is always mediated through the individual's cognitive representations of the outside world. The knowledge and cognitive processes thought to reside in the mind of the individual cannot be directly observed, but it is possible to hypothesize what they might be like in terms of the information they contain and how that information might be structured. Working from these assumptions, cognitive psychologists have put considerable effort into describing, sometimes in the form of computer programs, knowledge structures and processes they hypothesize to underlie competent performance on various mathematical tasks (Greeno, 1987b; Resnick, 1985).

Greeno (1987b) has referred to this general program of research as the knowledge structure program. The intent is to make explicit knowledge that is often implicit, but that is required for competent mathematical performance. Knowing mathematics from this perspective means having in place the knowledge and cognitive processes needed to carry out various mathematical tasks. An important role of this research is to specify in detail what that knowledge is, especially the knowledge that is implicit. This implicit knowledge made explicit can be viewed as revealing the knowledge underlying understanding in the domain. The resulting models might even be used to couch objectives of instruction in terms of desired cognitive structures instead of behavioral outcomes (Greeno, 1980; 1987a).

In building models of the knowledge underlying mathematical performance, cognitive researchers have relied on two kinds of analysis: (a) detailed analysis of students performing mathematical tasks, both correctly and incorrectly and (b) detailed analysis of the mathematical content involved in the task. The work has revealed important aspects of knowledge that had formerly remained implicit and has revealed the complexity of knowledge required to perform seemingly simple tasks. The research has also resulted in rich descriptions of how children solve problems in various mathematical domains taught in school. These include descriptions of the kinds of errors or incorrect applications students make, but also include the various correct or appropriate procedures children use and invent to solve various tasks. Some of the work specifies typical developmental sequences in which students progress through various strategies or procedures in a domain (e.g., Carpenter & Moser, 1982; Fuson, 1982).

Domains that have been studied from the perspective of the knowledge structure program include addition and subtraction (e.g., Carpenter & Moser, 1982; Riley, Greeno, & Heller, 1983; Vergnaud, 1982), rational numbers and fractions (e.g., Behr, Lesh, Post, & Silver, 1983), and decimal fractions (e.g., Resnick et al., 1989). In addition to these topics, many researchers have focused on the knowledge and skill involved in problem solving, both in general and in specific domains. Rather than attempt a comprehensive review of research in these domains, we will offer here a few significant examples that illustrate important implications for thinking about learning and knowing mathematics in elementary schools. We begin with the knowledge structures hypothesized to underlie competent solving of addition and subtraction word problems. We then turn to computational skills in arithmetic and, finally, to problem solving.

Schemata for word problems. A number of cognitive researchers have directed their efforts at describing the knowledge underlying tasks involving addition and subtraction of whole numbers (see, e.g., Carpenter, Moser, & Romberg, 1982). In part this emphasis has been due to the relative simplicity of the domain; although detailed analyses have shown that addition and subtraction are much more complex that they seem on first glance, they are not as complex as other domains in school mathematics, such as rational numbers or multiplication and division. The work has also merged with research examining the development of children's knowledge and skill in counting and counting-based strategies for adding and

subtracting (Fuson, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983), resulting in a rich body of research on addition and subtraction. Researchers studying addition and subtraction word problems have emphasized the structure of such problems and have drawn heavily on schema and related constructs arising from cognitive research on reading comprehension (Anderson, 1984; Mandler, 1984). Word problems are viewed from this perspective as a special kind of text to be comprehended.

Reading research has pointed to the powerful influence of prior knowledge on the comprehension of text (Anderson, 1984; Mandler, 1984). In contrast to earlier views, cognitive scientists generally agree that comprehension is not a matter of somehow absorbing or recording information inherent in written or spoken language. Because all written and spoken language is in some sense incomplete, the reader draws heavily on his or her prior knowledge and expectations in building a representation of the situation described by the text. Schema theory (Anderson, 1984; Mandler, 1984) holds that schemata are forms of knowledge that play a critical role in this constructive comprehension process. Schemata are prototypical versions of situations or events that are stored in long-term memory and built up over many experiences with those situations. Schemata provide a framework within which to interpret text; comprehension is impossible without an appropriate schema. An example of how the schemata that a reader brings to text can affect comprehension is offered in a study (Anderson, Reynolds, Schallert, & Goetz, 1977) in which music students interpreted the following passage as describing an evening of playing music together whereas physical education students interpreted it as being about playing cards.

> Every Saturday night, four good friends get together. When Jerry, Mike, and Pat arrived, Karen was sitting in her living room writing some notes. She quickly gathered the cards and stood up to greet her friends at the door. They followed her into the living-room, but as usual they couldn't agree on exactly what to play. Jerry eventually took a stand and set things up. Finally, they began to play. Karen's recorder filled the room with soft and pleasant music. Early in the evening, Mike noticed Pat's hand and the many diamonds. As the night progressed, the tempo of play increased. Finally, a lull in the activities occurred. Taking advantage of this, Jerry pondered the arrangement in front of him. Mike interrupted Jerry's reverie and said, "Let's hear the score." They listened carefully and commented on their performance. When the comments were all heard, exhausted but happy, Karen's friends went home. (p. 372)

Some researchers have taken the view that the comprehension of mathematics word problems, as a special kind of text, similarly requires the reader to bring to bear appropriate knowledge about quantities and relationships among quantities in the form of schemata (Briars & Larkin, 1984; Kintsch & Greeno, 1985; Mayer, 1982; Riley et al., 1983). For example, being able to solve the following word problem

requires recognizing the relationships among the known and unknown quantities involved; it is not enough to have learned associations between particular words and operations (e.g., altogether means to add):

> Jim had 10 marbles. Bob gave him some more marbles. Then Jim had 13 marbles altogether. How many marbles did Bob give to Jim?

To support this view, cognitive researchers have used the strategy of building computer models that make explicit the tacit knowledge involved in understanding and solving addition and subtraction word problems (Briars & Larkin, 1984; Riley et al., 1983). This research is clearly an example of the general strategy of describing the knowledge structures hypothesized to underlie mathematical performance. As Greeno (1987a) put it,

> A cognitive model of understanding and solving problems simulates the process of understanding by constructing representations based on the words in problem texts. The representations contain information that students appear to gather from the texts and use in their solutions. The process can be characterized as the recognition of patterns of information. (p. 63)

The models developed by Riley et al. (1983) and by Briars and Larkin (1984) are grounded in extensive empirical work by various researchers to describe the strategies children use to solve simple addition and subtraction problems and in careful analysis of the kinds of word problems that require addition and subtraction (Carpenter & Moser, 1982; Vergnaud, 1982). These analyses have shown that there are important patterns of relationships among quantities in addition and subtraction word problems that are not typically addressed in instruction. Most characterizations of these relationships are based on the three patterns in Table 1. Successful solvers of such problems have knowledge of these patterns, but the knowledge is tacit.

Building the computer models requires representing this implicit knowledge explicitly, making it available for examination. For example, the model developed by Riley et al. (1983) uses the schema represented in Figure 1 to solve Problem 2 in Table 1. When encountering Problem 2, the computer model places the information from the problem into the various slots of the schema. It then uses a series of rules for operating on these organized quantities to produce the missing element, the answer to the problem. Riley et al. built a series of such models that, with the increasing structure and amount of

Table 1

Types of Addition and Subtraction Word Problems

Type		
Change	there is an event in which an initial quantity is increased or decreased	1. "Connie had 5 marbles. Jim gave 8 more marbles. How many marbles does Connie have altogether?"[a]
		2. "Joe had 8 marbles. Then he gave 5 marbles to Tom. How many marbles does Joe have now?"[b]
Combine	There are two individual quantities that are not changed but thought of in combination	3. "Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?"[a]
		4. "Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?"[a]
Compare	there are two individual quantities to be compared	5. "Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?"[a]
		6. "Jim has 5 marbles. He has 8 fewer marbles than Connie. How many marbles does Connie have?"[a]

[a]Problems taken from Carpenter and Moser (1983, p. 160).

[b]Problem taken from Riley, Greeno, and Heller (1983).

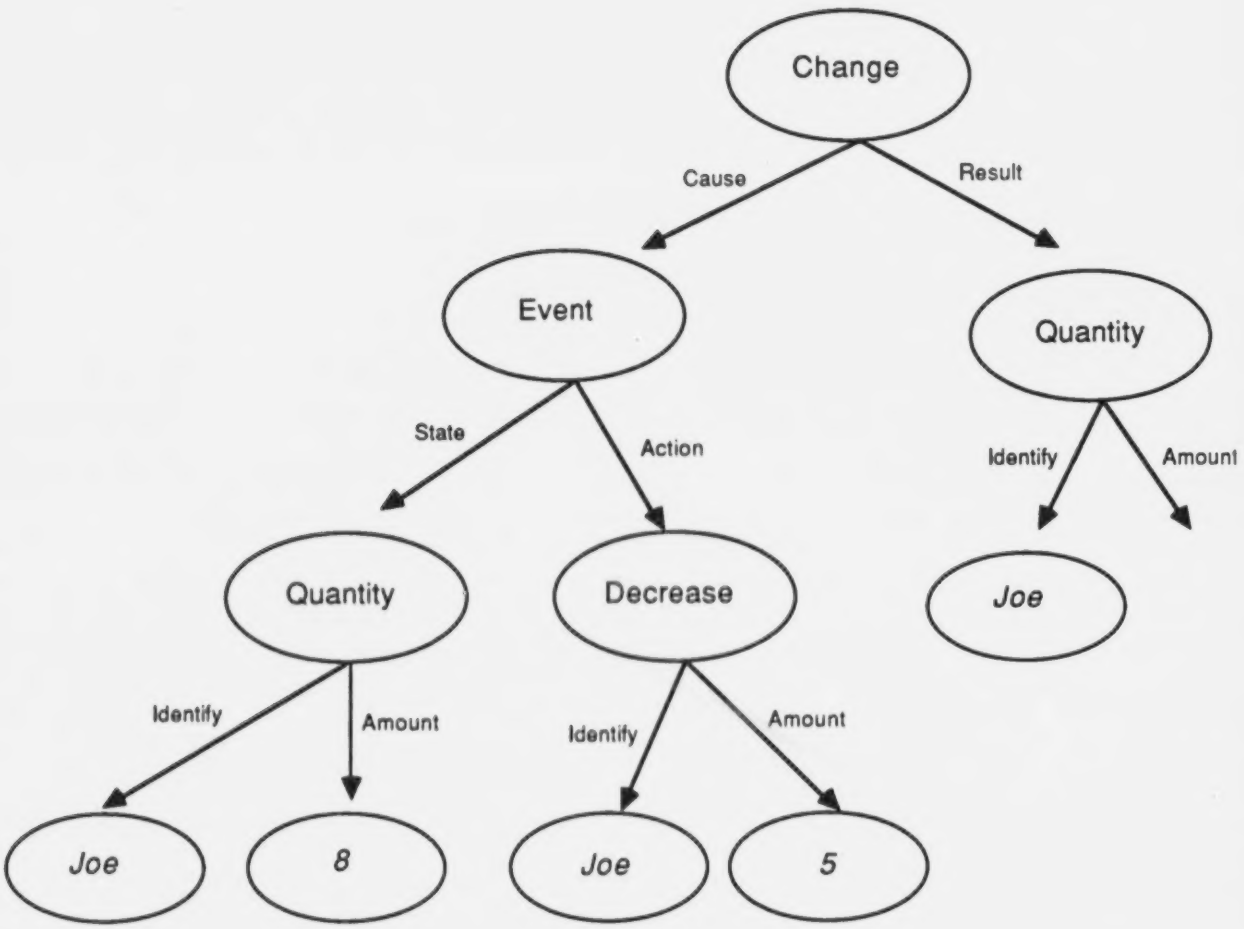

Figure 1. Schema for change situation (adapted from Riley et al., 1983).

information that can be represented, parallels the development in children of the ability to solve these kinds of problems. A model developed by Briars and Larkin (1984) to solve the same kinds of problems makes less use of the explicit schematic representations, building more of the knowledge of patterns into rules for operating on the quantities in the problem. The key feature of both models, though, is that a crucial aspect of solving these word problems is recognizing the patterns of quantities in the problems. In essence, recognizing these patterns is what it means from this perspective to understand the problem. It is not enough to have knowledge of simple correspondences between individual words such as altogether, gave, or less and mathematical operations such as addition and subtraction. Rather it is having available schemata for grasping the relationships among the entire set of quantities involved that permits understanding and solution of the problem. For thinking about mathematics in elementary schools, this research provides a detailed characterization of knowledge hypothesized to be important for solving textbook word problems. Also implicit in this work is the assumption that the schemata that provide frameworks for solving these word problems would also provide the problem solver with frameworks for thinking about problems not encountered in written form, for example, figuring out how many marbles have been won or lost in an actual marble game. The instructional implications that follow from these descriptions of existing school tasks are not automatic. One way of using these descriptions is as a set of instructional objectives described in cognitive rather than behavioral terms. Thus, given the instructional goal of being able to solve these sorts of problems, this research can be viewed as describing the knowledge that students need to acquire. Another possibility is to design instructional representations that make explicit the implicit knowledge revealed by these analyses (Greeno, 1987a). In research described later in this paper, Carpenter, Fennema, and Peterson (1987) have considered the knowledge of problem structures and children's solution strategies resulting from this work to be important knowledge for teachers to have in helping children learn addition and subtraction.

Computational skill. Another domain that has been the object of cognitive scientists' analyses is computational skill in arithmetic. Although its central role is being questioned in most current calls for reform, computation pervades the traditional elementary school mathematics curriculum. In the high school curriculum there is a parallel focus on symbol manipulation skills in algebra. Cognitive researchers

have worked to develop models of the knowledge structures underlying these ubiquitous skills and to explain the errors students make. One motivation for this research has been to use computation as a site for exploring more general issues about how people learn and know procedures (e.g., Brown & Burton, 1978). Another motivation has been to understand better the nature of computational skill and its role in knowing and understanding mathematics.

Researchers in mathematics education have long documented the kinds of errors students make in computational tasks (Ashlock, 1982; Buswell, 1926). Tools from cognitive science for representing procedural knowledge, such as production systems (Newell & Simon, 1972), have enabled cognitive researchers to conjecture with greater precision about the knowledge underlying computational skill and to develop models to explain why students make various kinds of errors. The best known example of this work deals with students carrying out of the traditional subtraction algorithm involving regrouping, or borrowing. By analyzing the errors made in the subtraction algorithm as carried out by hundreds of students, Brown and Burton (1978) developed a computer model of the procedural knowledge underlying correct and erroneous computational performance. Their model considered the computational errors made by students to result from faulty rules. Like bugs in computer programs, these buggy algorithms resulted in incorrect but rule-governed performance. Thus, students' errors were seen to result, not from an absence of knowledge but from the application of faulty knowledge in the form of incorrect procedural rules. A similar account of errors in algebra has been offered by Sleeman (1982).

At one level, these analyses of computational errors simply provide highly detailed specification of student performance in the form of correct and faulty rules. But they also represent a view of procedural knowledge as consisting of organized sets of rules, both correct and incorrect, that individuals have learned. To know mathematics from this perspective is to have acquired procedural rules for manipulating the written symbols of arithmetic. But where do children learn these incorrect procedures that are obviously not directly taught in school? Brown and VanLehn (1980; see also VanLehn, 1983) argue that children infer or invent these faulty procedures from the partial procedural knowledge they have when they reach points at which they do not know what to do next in carrying out a procedure, for example, when confronted with the need to subtract a larger digit from a smaller one. What is significant about this

characterization of knowledge is that it deals almost solely with features of the written symbols of arithmetic, rather than the quantities represented by those symbols or principles governing those quantities (Resnick, 1982). The fact that these theories do such a good job explaining the computational errors that students actually make suggests that students' computational knowledge may, indeed, be organized primarily around visual cues and the physical arrangement of symbols, not around the underlying quantities to which the symbols refer (Davis, 1984; Resnick, 1982). The view of mathematical knowledge suggested here is one of structured procedural rules that operate on written symbols.

Problem solving. Problem solving is a part of the mathematics curriculum that has been a long-standing concern for mathematics educators and has been the focus of research on knowledge structures. Most calls for reform place the ability to solve problems at the center of desired outcomes of schooling (National Council of Teachers of Mathematics, 1989; National Research Council, 1989). Problem solving has also been held up as a desirable means of learning mathematics, but our focus here will be on cognitive research that examines the ways individuals solve problems and that seeks to characterize the knowledge structures underlying successful problem solving, hence our inclusion of problem solving as part of the more general theme of the importance of knowledge structures. But problem solving, like representation, permeates most discussions of mathematics knowing and learning, and so will reappear throughout the paper. In particular, we later consider problem solving from the perspective of the discipline of mathematics. Here, however, our focus is on attempts by cognitive researchers to characterize the knowledge structures and processes underlying successful problem solving.

Most current theories of problem solving are based on an information-processing model of human thinking and draw heavily on early work in this tradition (e.g., Newell & Simon, 1972; Simon, 1978). Because much of this early work was directed at describing general processes of problem solving, it focused on tasks that minimized the role of an individual's knowledge in the problem-solving process (Simon, 1978). So researchers used puzzle-like tasks such as the Tower of Hanoi, which involves moving a set of concentric disks on pegs according to certain rules, or the familiar missionaries and cannibals problem, which involves moving an equal number of missionaries and cannibals across a river in a boat with

the constraint that missionaries can never be outnumbered by cannibals. These tasks require mostly knowledge that could be provided when presenting the problem. The problems were also highly structured in that permissible moves were carefully specified and the goal of the problem was clearly defined.

Central to Newell and Simon's (1972) theory of problem solving is the mental representation of the problem that the individual problem solver creates in working memory. This problem space contains the individual's mental representation of the information in the problem and permissible moves to be used in solving it. In constructing this problem space the problem solver draws both upon the problem as presented and upon knowledge represented in long-term memory that can be brought to bear in solving the problem. The notion of problem space brings into sharp focus the assumption in this research that thinking and knowing take place in the mind: In solving a problem, the problem solver is viewed as manipulating representations or symbols in the mind. To the extent that objects in the external environment come into play, it is the representations of these objects that the problem solver manipulates. It is this assumption of knowing and thinking being completely internal to the individual mind that is being questioned by researchers who are emphasizing the situated nature of cognition.

When focusing on structured, knowledge-lean tasks, Newell and Simon (1972) emphasized a model of problem solving as a search through the permissible moves represented in the individual's problem space. This resulted in a focus on the general strategies or heuristics that individuals use for conducting these searches in the solving of various problems. An example of such a general strategy is means-ends analysis, in which the problem solver considers the desired goal state (e.g., getting all the missionaries and cannibals to the other side of the river) and considers possible moves that will bring the representation of the problem in the problem space closer to that goal (e.g., getting one more cannibal to the other side). This sort of means-ends analysis is a general strategy that will work in a variety of contexts.

From this perspective the important knowledge to have for good mathematical thinking and problem solving was a repertoire of general processes or strategies. Clearly a person had also to have mathematical knowledge for these general strategies to operate on, but the emphasis was on the general processes. This emphasis on general strategies and processes was consistent with and gave support to

the many attempts to train students in general strategies for problem solving, critical thinking, and other forms of "higher-order" thinking (for reviews, see Chipman, Segal, & Glaser, 1985; Nickerson, Perkins, & Smith, 1985; Resnick, 1987a; Segal, Chipman, & Glaser, 1985). In mathematics, most attempts to teach general problem-solving strategies stem from George Pólya's (1957) classic characterization of the problem-solving process as involving heuristics such as finding simpler problems, using diagrams, and considering special cases. These sorts of general rules of thumb pervade many programs for teaching problem solving and the treatment of problem-solving in elementary-school mathematics textbooks.[2]

But researchers began to question the power of general strategies as central for problem solving and understanding. As Schoenfeld (1985; 1987) has pointed out, although heuristics like those proposed by Pólya are good descriptions of what successful problem solvers do, they are not specified with enough detail to be prescriptive--to help others learn to carry them out. Thus attempts to teach these heuristics have generally not been successful. In addition, as researchers extended the general information-processing model to study problem solving in information-rich domains such as physics and mathematics, the importance of specific knowledge available in long-term memory came to the fore (Chi, Glaser, & Rees, 1982; Glaser, 1984; Resnick, 1987a). The use of general strategies like means-ends analysis did not distinguish the performance of expert problem solvers in these domains from that of novices. Rather it was the experts' rich store of organized accessible knowledge and ways of representing problems that characterized their successful performance.

One important role played by this domain-specific knowledge is in how the problem solver represents a problem to be solved. For example, in one study (Larkin, McDermott, Simon, & Simon, 1980), physics novices (undergraduates who had completed a single physics courses) tended to solve problems by selecting as quickly as possible formulas and equations into which values in the problem could be placed and calculated. In contrast, experts (graduate students in physics) worked to build a representation of the entire problem, usually structured around general constructs and principles in physics. Only then did the

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[2]We discuss in more detail Pólya's views about the role of heuristics when we consider problem solving from the perspective of mathematics.

experts move to formulas and equations, often after they had virtually solved the problem by using more qualitative representations.

Further evidence that experts construct different mental representations of problems than do novices is offered by studies in which experts and novices classify various sorts of problems. For example, when Chi, Feltovich, and Glaser (1981) had individuals sort physics textbook problems, novices sorted the problems on the basis of surface features such as the kinds of objects involved in the problem (e.g., levers, pulleys, or balance beams) or similarities in the diagrams presented with the problem. Experts classified the problems according to the physics principles that were needed to solve the problems, (e.g., conservation of energy), suggesting that they had ways of representing the problems that were not readily available to the novices. Silver (1979) similarly found that unsuccessful solvers of mathematics word problems were more likely to rely on surface features when categorizing word problems than successful problem solvers, who relied on similarities in underlying mathematical structure.

Thus, researchers studying problem solving came to focus on the domain-specific knowledge that the problem solver has available and how that knowledge is organized (Glaser, 1984). In mathematics, this focus was reflected in the research we discussed earlier on schemata for solving addition and subtraction word problems and in the argument that successful problem solving involves being fluent with a repertoire of representation systems that can be used in problem solving (Kaput, 1988; Lesh et al., 1987). Both of these views of understanding and problem solving emphasize the importance of having the appropriately structured domain-specific knowledge over knowledge of general processes or strategies. Similarly, as an alternative to the general heuristics suggested by Pólya (1957), Schoenfeld (1985, 1987) focused on specifying the strategies used by successful problem solvers at a level of detail that included more of the mathematics knowledge involved. For example, Schoenfeld found that heuristics such as "examine special cases" actually comprised multiple more specific strategies, such as "If there is an integer parameter n in a problem statement, consider the values $n = 1, 2, 3, 4, \ldots$ You may see a pattern that suggests an answer, and the calculations themselves may suggest the mechanism for an inductive proof that the answer is correct" (Schoenfeld, 1987, p. 19). Schoenfeld (1985) found that instruction in his own problem-solving course for undergraduate college students based on these detailed strategies was

successful in helping students solve both problems like those used during instruction and novel problems. He also emphasized the importance of metacognitive knowledge and beliefs about mathematics for successful problem solving.

Debate over the relationship between general problem-solving and thinking skills, on the one hand, and domain-specific knowledge on the other, continues (e.g., Glaser, 1984; Perkins & Salomon, 1989; Resnick, 1987a). Any resolution of the debate will likely involve some balance between the two (Perkins & Salomon, 1989). What is common to all these views of problem solving is that they attribute success in problem solving essentially to the problem solver having acquired appropriate knowledge in the form of general strategies or organized domain-specific knowledge.

Summary of knowledge structures. We have offered three examples of domains in which cognitive researchers have worked to describe in some detail the knowledge hypothesized to underlie mathematical performance: addition and subtraction word problems, arithmetic computation, and problem solving. These and other similar analyses have resulted in a rich understanding of some domains in the existing mathematics curriculum. Inherent in these analyses is the assumption that we can understand knowledge, thinking, and understanding in mathematics by specifying knowledge structures underlying competent performance. For solving addition and subtraction word problems, these knowledge structures included schema-based knowledge of the quantitative structures in the situations described by the story problems. For computation, the hypothesized knowledge structures consisted of procedural rules that operate on the written symbols of arithmetic. For problem solving, different kinds of knowledge have been emphasized as important at various times by researchers, including knowledge of general strategies and heuristics, organized and rich domain-specific knowledge of mathematics, and metacognitive knowledge. In all these cases, knowing mathematics means having acquired appropriate knowledge structures.

The knowledge structure program has focused primarily on describing the what of knowing mathematics: What is the knowledge that underlies competent mathematical performance? The lens with which cognitive researchers address this question is a powerful one that can reveal hitherto unexamined aspects of mathematical knowledge. At the same time this research program is built on a fairly narrow view

of what it means to know mathematics. Most of the tasks subjected to this cognitive analysis have been taken from the traditional school mathematics curriculum and thus are descriptions of how individuals understand and perform mathematics under current instructional practices. The research does not address the issue of whether these traditional school tasks are worthwhile.

Research in the knowledge structure program has resulted, for the most part, in theories about how people think and what their knowledge might be like, not about how to best help individuals acquire that knowledge. As Greeno (1987a) put it,

> Cognitive theory provides hypotheses about the knowledge and skill of successful student problem solvers and the ways in which their knowledge and skill differ from those of less successful ones. . . . Although cognitive models can describe more or less accurately the knowledge and skill we want students to acquire, the experiences that will help students acquire that knowledge and skill constitutes a separate issue. (p. 69-70)

These researchers have examined the performance of both competent and novice individuals, but they have generally not in this program of research looked at changes in knowledge as students learn or develop expertise in a domain.

Understanding as Connections Among Types of Knowledge

In describing the kinds of knowledge that constitute or underlie understanding of mathematics, many researchers emphasize connections among various kinds of knowledge. In part, this emphasis is a reaction to behaviorist and associationist learning theories and school curricula that present mathematical knowledge as collections of relatively isolated concepts, rules, and procedures to be learned. Mathematics educators have long expressed concern over the learning of the symbols and procedures of mathematics as rote learning, devoid of understanding (Buswell, 1926; Davis, 1986). Most mathematics educators and research agree that it is possible to learn many of the symbols of mathematics and procedures for computing and manipulating those symbols without also learning much accompanying understanding of the quantities or mathematical entities represented by the symbols and without acquiring the knowledge needed to use the skills when needed. There is less agreement about the kinds of links or connections that need to be established in the mind of the learner to constitute the desired understanding. Some researchers have emphasized the distinction between procedural and conceptual

knowledge of mathematics, and have sought to characterize the relationship between them in various ways. Others have called for more connections between the formal, symbolic mathematics learned in school with the rich base of informal knowledge children develop in out-of-school settings.

Conceptual and procedural knowledge. Some cognitive psychologists distinguish between knowledge and understanding of concepts of mathematics on the one hand, and knowledge of the procedures of mathematics on the other (Hiebert, 1986; Nesher, 1986). Researchers vary, of course, in their definitions of these kinds of knowledge. Hiebert and Lefevre (1986) defined conceptual knowledge as "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (pp. 3-4). From this perspective, terms like understanding or meaningful learning essentially refer to knowledge that is highly interconnected through relationships at various levels of abstraction.

Procedural knowledge, according to Hiebert and Lefevre (1986), consists of knowledge of (a) the formal symbol system of mathematics and (b) "rules, algorithms, or procedures used to solve mathematical tasks" (p. 6). The first part is knowledge of the conventional forms in which mathematical ideas are expressed, including for example, the ability to recognize that 5 + 6 = 11 is an acceptable form, whereas 5 + = 6 is not. The second part of procedural knowledge consists of instructions for completing various tasks. These procedures may operate primarily on standard written symbols, as is the case for the algorithms for multiplication and division, or they may operate on concrete objects, or other objects that are not the standard symbols.

Elementary school mathematics instruction typically emphasizes the learning of procedures applied to standard written symbols in ways that leave them unconnected to conceptual knowledge. Students thus learn the procedures and symbols as meaningless marks on paper. From Hiebert and Lefevre's perspective, written mathematical symbols take on their conventional mathematical meanings by being linked to conceptual knowledge of ideas encountered through experience. For example, if the idea of change as in Problem 1 in Table 1 is linked to "+," that symbol takes on meaning. Hiebert and Lefevre (1986) argued that "if students connect the symbols with conceptually based referents, the symbols

acquire meaning and become powerful tools for recording and communicating mathematical events" (p. 20).

Nesher (1986) argued that conceptual knowledge should be thought of as the control structure for procedural, or algorithmic, knowledge. A person can carry out procedures without much conscious thought by relying on physical feedback as long as things are going well. When something goes awry, however, the learned procedures may not work, and the person must mentally step back and take stock of the situation. This is where conceptual understanding plays a role. An important consequence of this view of conceptual knowledge as control structure is that it cannot exist without at least some procedural knowledge. Since the conceptual knowledge is, in essence, knowledge about the procedures, it can be developed only by reflecting, in part, on the procedures themselves, which must therefore be learned before, or at least in tandem with, the conceptual knowledge. Nesher gives the example of trying to help college students learn the concept of arithmetic mean in an introductory statistics class and needing to have students compute some means before being able to talk about the concept of mean with any meaning. Nesher pointed out that there is little solid evidence for the apparent belief held by many mathematics educators that solid conceptual knowledge of a topic will produce correct procedures as a natural consequence. Indeed, in at least some cases, procedural knowledge must form the basis for conceptual understanding.

Another hypothesis about the relationship between conceptual and procedural knowledge in the domain of young children's counting has been offered by Gelman and her colleagues (Gelman & Meck, 1983; Gelman & Meck, 1986; Greeno, Riley, & Gelman, 1984). They argue that conceptual competence consists of implicit knowledge of principles that constrain but do not determine procedural performance. These principles are much like the principles of grammar that constrain the utterances we make; we use these principles without being aware of them. Conceptual competence for counting includes such principles as one-to-one correspondence and the cardinality principle (that the last number recited is also the number of objects in the counted set). Knowledge of these principles provides guidance and constraints for the procedures to be used in a particular counting situation. Every counting task is a bit different, and a child that is able to count appropriately in a wide variety of settings evidences an

understanding of basic principles underlying counting, rather than the learning of a single counting procedure.

Greeno, Riley, and Gelman (1984) hypothesized that the actual procedure of counting in a particular situation is generated on the basis of the principles that constitute conceptual competence, with procedural competence providing a set of tools for transforming the principles of conceptual competence into procedures, and utilizational competence providing knowledge of the mappings of these procedures to situations. There are several points from this work that are important for our consideration of the relationship between conceptual and procedural knowledge. First, knowledge of the principles that form conceptual competence are implicitly rather than explicitly known. Second, successful performance in a variety of settings can be taken as evidence for an implicit understanding of the underlying principles. Third, having knowledge of correct principles does not guarantee being able to carry out procedures correctly or to generate appropriate procedures in new situations; the procedural or utilizational competence needed to generate the procedure in a particular situation may not be available. Finally, conceptual knowledge serves to constrain performance but not to determine it completely, an idea similar to Nesher's (1986) control structure. A similar idea is that conceptual knowledge serves as a set of critics for procedures generated (Resnick, 1982; VanLehn, 1983).

Formal and informal knowledge. Some researchers have emphasized the importance of having connections between knowledge of the formal, symbolic mathematics taught in school and the informal, intuitive knowledge of mathematics gained from everyday experience. Ginsburg (1977) and Resnick (1986; 1987b) have argued that a major reason for the difficulty students have in learning the formal, symbolic mathematics taught in school is that this formal knowledge does not get linked to their rich informal knowledge base derived from working with quantities in everyday situations. Many children appear to view school mathematics as a collection of arbitrary rules and procedures performed on meaningless symbols, in spite of the fact that they may have developed rather sophisticated concepts and strategies for solving quantitative problems encountered out of school. The informal strategies that both children and adults construct often reveal understanding of the kind of mathematical principles that

Greeno et al. (1984) referred to as conceptual competence, but students do not seem to draw upon this competence in the learning and doing of the procedures of school mathematics.

An example of competent informal strategies developed outside of school is offered by a series of studies of Brazilian children without much formal schooling (Carraher, Carraher, & Schliemann, 1983). Working as street vendors, these children developed considerable proficiency at mental arithmetic strategies for figuring prices for customers, as illustrated in the following:

> How much is one coconut? *Thirty-five.* I'd like 10. How much is that? [Pause] *Three will be 105; with three more, that will be 210.* [Pause] *I need four more. . .* [Pause] *315. . . I think it is 350.*

This child solved, by using repeated addition, a problem that most schooled adults would solve by using multiplication. The child used the memorized price of three coconuts to reduce the number of additions required, demonstrating implicit understanding of the important principle of additive composition--the idea that numbers can be additively broken apart and recombined (Resnick, 1986). The children studied by Carraher et al. (1983) could solve a variety of other problems dealing with figuring prices by using various invented mental arithmetic strategies. When presented with the same problems in written form, however, the children tried unsuccessfully to apply the algorithms they had learned in their short stays at school. They seemed unaware that they could apply the informal strategies they used every day to these "school" tasks.

When school tasks are set in a way to encourage it, children can use their informally invented strategies to solve problems that they could not solve using the school taught algorithms (Carpenter & Moser, 1982; Ginsburg, 1977). Many of these informal strategies are built on the extensive use of counting and the additive relationships, as in the Brazilian example. But traditional school instruction does not seem to help students connect this informal competence to the formal symbols and symbol rules that are the focus of instruction.

Researchers calling for the building of better connections between informal mathematical knowledge and formal school knowledge of mathematics generally base their arguments on two assumptions about the nature of mathematical understanding. The first is that children's informal knowledge can serve as a powerful base upon which to build more formal knowledge of mathematics. The informal knowledge that

seems to be rooted in basic principles such as additive composition can provide meaningful referents for the symbols of formal arithmetic. Thus the informal knowledge can bring meaning to the formal mathematics. Second, by linking students' knowledge of formal school arithmetic to their informal knowledge based in experience, students will be in a better position to apply their more formal knowledge to the solving of problems encountered in out-of-school settings. Another way to consider these connections between formal and informal knowledge is as the need for transfer in both of these directions (Pea, 1988). Both of these arguments are closely related to the constructivist perspective, with its emphasis on the meaningful learning of mathematics taking place through modifying and building upon existing knowledge and ways of thinking.

Learning as Active Construction of Knowledge

Central to virtually all cognitive theories is the assumption that individuals interpret their environments through existing cognitive structures built up through adaptation to the environment (Resnick, 1983; Shuell, 1986). This constructivist perspective has had a profound influence on how many mathematics educators think about understanding and learning mathematics (Kilpatrick, 1987), especially the idea that for instruction to foster learning with meaning or understanding, it must somehow attend to and build upon children's existing knowledge.

The active view of the learner in cognitive psychology contrasts with earlier behaviorist and associationist perspectives, in which a students' knowledge could be viewed as a sort of cumulative record of his or her experiences. Although the learner was considered to be *active* in behaviorist theories, it was in a different sense of *active* than in current cognitive perspectives. Behaviorists argued that learning takes place only when the individual overtly responds to environmental stimuli; an individual learns only those behaviors that are actually carried out and reinforced (Skinner, 1986). Indeed, this sort of active response provides the foundation for theories of operant conditioning. From this behaviorist perspective, an individual's learning is determined by the responses he or she makes to environmental stimuli; thus learning can be made more efficient by carefully structuring those environmental stimuli so

that the learner makes responses that are gradually shaped toward the target behavior. The most efficient learning is errorless learning, with the learner's responses becoming increasingly refined.

When cognitive theorists refer to the learner as being active, they mean something quite different than overt responding. They mean that the learner plays an active role in interpreting and structuring environmental stimuli. Rather than passively receiving and recording incoming information, the learner actively interprets and imposes meaning through the lenses of his or her existing knowledge structures, working to make sense of the world. At the same time, learning or development takes place, not by the simple reception of information from the environment, but through the modification and building up of the individual's knowledge structures.

An important result of this shift in perspective has been a blurring by cognitively oriented researchers of the distinction between learning and cognitive development. When learning was considered to be acquisition or absorption of knowledge, development of fundamental cognitive structures, such as those supporting basic logical and quantitative thinking (Piaget, 1983) could be considered a separate domain of inquiry. With learning viewed as the active integration by the learner of new information with existing knowledge, the line between learning and development can no longer be drawn cleanly. Neither can a clear line be drawn between "natural" mathematical thinking and the use of cognitive tools available in the environment. It is important to note that analyses of how children develop or acquire mathematical knowledge and skill in specific domains (for example, the development of children's early conceptions of number and counting--Carpenter & Moser, 1982; Fuson, 1988; Gelman & Gallistel, 1978; Steffe et al., 1983), are studies of how children's conceptions and skills develop in current cultural and instructional environments. Although the various sequences of acquiring mathematical knowledge derived from this research are sometimes assumed to be "natural," they are actually the result of experience in particular environments.

Research on children's mathematical knowledge and skill provides ample evidence that they do, indeed, construct or invent new knowledge on the basis of what they already know (Resnick, 1985). Our first example comes from research on children's use of various counting strategies to solve simple addition and subtraction problems. Researchers have found that children use a variety of counting strategies to

solve problems of the form a + b = ?, where a and b are whole numbers between 1 and 10 (Carpenter & Moser, 1982; Fuson, 1988). Of particular interest here are the ALL strategy and the MIN strategy. In using the ALL (or counting all) strategy, the child first counts to a, then counts b more units to arrive at the sum. For example, to add 3 + 6 = ?, the child would count, "1, 2, 3," and then, "4, 5, 6, 7, 8, 9." The counting could be done with objects, with fingers, or mentally. The more efficient MIN (or counting on from larger) strategy involves beginning with the larger of the two addends and counting on for the smaller addend. To add 3 + 6 = ?, the child would count, "6," then "7, 8, 9." Many children come to use the MIN strategy even though it is not generally taught directly (Carpenter & Moser, 1982; Fuson, 1982). In a carefully controlled study, Groen and Resnick (1977) found that a number of kindergarten children whose instruction focused exclusively on the ALL strategy invented and used the MIN strategy to solve addition problems.

Use of the MIN strategy is an example of an appropriate, or correct, invention based on existing knowledge and instruction. The buggy algorithms for subtraction we described earlier provide examples of the invention of incorrect procedures. Brown and VanLehn (1980; see also VanLehn, 1983) posit in their repair theory that the errors students make in carrying out the computational procedures are the result of inappropriately invented or repaired procedures. When a student reaches a point in carrying out the procedure at which he or she does not know what to do next, a repair or patch is made, often resulting in a computational error. These repairs can be thought of as on-the-spot invented procedures based on the student's existing knowledge. As we pointed out earlier, Resnick (1982) has argued that the repairs children make are quite reasonable inventions if only knowledge of the syntax, or surface features of the symbols is taken into account. For example, children usually end up putting digits in each column of the answer and "borrow" marks at the top of the problem. The repairs are not reasonable, however, if one takes the meaning, or semantics of the symbols into account. Children making these errors violate such principles as the maintaining of the quantities represented by each number in the problem. So it seems that children are making reasonable inventions; they are just failing to base these inventions on the appropriate knowledge.

Thus, it is clear that children do not simply absorb mathematical knowledge as it is presented, but impose their existing frameworks of knowledge to incorporate and invent new knowledge. Virtually all cognitive researchers and mathematics educators ascribe to this constructive view of learning and knowledge. Some theorists and researchers take a stronger position on the constructed nature of knowledge and learning (e.g., Steffe & Cobb, 1988; von Glasersfeld, 1987). Because these radical constructivists form an important voice in the mathematics education community, we consider their views here in some detail.

Radical constructivism. Radical constructivists emphasize the epistemological assumption that the world external to the individual cannot be known in any ultimate sense, but that all knowledge is a cognitive construction of the individual. Von Glasersfeld (1989) has identified two principles of constructivism as a theory of knowledge: "(a) knowledge is not passively received but actively built up by the cognizing subject; and (b) the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (p. 162). It is the first principle that is embraced by most cognitively oriented psychologists and mathematics educators when they argue that knowledge is actively constructed (Kilpatrick, 1987). It is adherence to the second principle that sets apart the radical constructivists. This second principle rejects the realist or empiricist assumption that the world can be known in any objective or ultimate way. Rather, physical reality can only constrain the cognitive constructions that individuals make. As von Glasersfeld (1988) put it, "...knowledge cannot aim at 'truth' in the traditional sense but concerns the construction of paths of action and thinking that an unfathomable 'reality' leaves open for us to tread" (p. 2). Sinclair (1988), writing from a Piagetian perspective, similarly argued that, "at all levels the subject constructs 'theories' (in action or thought) to make sense of his experience; as long as these theories work the subject will abide by them" (p. 29). But these personal theories will always be only approximations to reality; external reality will always possess properties unknown to the individual (Piaget, 1980; von Glasersfeld, 1988). Thus, knowledge and meaning are ultimately personal and, to some extent, idiosyncratic.

For radical constructivists, then, mathematical knowledge is exclusively a cognitive construction of the individual; there is no mathematical reality "out there" to be learned or discovered. Rather, mathematical

knowledge consists of "coordinated schemes of action and operation" (Steffe, 1988), ways of understanding and acting that have been built up by the individual. We experience a sense of "out thereness" of mathematical knowledge only because we impose our own conceptual organization on the world (Cobb, 1986). Note that this view of meaning differs from Hiebert's view, discussed earlier, that understanding of symbols resulted from the learning simply connecting or linking the symbol to external referents, implying that the meaning resided essentially in the physical objects.

Because mathematical knowledge exists only as constructed by the individual, it cannot be transmitted or instilled through communication (Cobb, 1988b; von Glasersfeld, 1988). In fact, communication itself is not a process of transmitting meaning, but of sending a set of instructions from which the recipient constructs a meaning. There is no meaning inherent in words, actions, or objects independent of an interpreter (Cobb, 1988b). Thus, a view of instruction as the transmission of knowledge is not acceptable to the radical constructivists: "the seemingly obvious assumption that the goal of instruction is to transmit knowledge to students stands in flat contradiction to the contention that students construct knowledge for themselves by restructuring their internal cognitive structures" (Cobb, 1988b, p. 87). Teachers cannot use language to "tell" or transmit knowledge to students, but "here and there to constrain and thus to guide the cognitive construction of the student" (von Glasersfeld, 1988).

As a result of this basic perspective, radical constructivist researchers in mathematics education have focused on describing and analyzing the "mathematics of children" (Steffe, 1988)--the meanings children place on their mathematical actions and the strategies that they have constructed through interaction with their home and school environments (see, e.g., Steffe, 1983). As we pointed out earlier, it is important to note that these environments are not inherently more "natural" than others. By focusing on knowledge as individual construction, constructivists create for themselves the difficult problem of explaining how people come to know and agree upon the same skills and understandings that constitute mathematics. How is it that "children come to know in a short time basic principles (in mathematics, but also in other scientific disciplines) which it took humanity thousands of years to construct?" (Sinclair, 1988, p. 1).

Increasingly, constructivists have dealt with this issue by focusing on the social nature of human interaction and its role in the individual construction of knowledge. Sinclair (1988), for example, pointed

out that Piaget, especially in his later writings (e.g., Piaget & Garcia, 1983), emphasized the role of society, with its accumulation of knowledge, on the individual's construction of knowledge. The objects with which children interact are defined largely by society. Adults in general, and teachers in particular, present children with real objects or with objects of thought in a certain way that makes it possible for them to re-discover or re-invent what it took their society a long time to elaborate.

> When Piaget, in his writings on education, asserted that "to understand is to invent or to discover," the inventions or discoveries are new to the child, but seen from the adult's point of view, they are re-creations. Our children do not have to invent the wheel: they can begin to conceptualize the intricate properties of wheels as they exist in our society. (Sinclair, 1988, p. 7)

Thus, teachers play an important role in presenting objects of thought to children. This presentation, along with the endogenous process of abstraction accounts for the learning of society's accumulated knowledge, but the presentation is often not optimal for the child's active construction of meaning (Sinclair, 1988).

Cobb (1988a) similarly argued that constructivists need to draw on an anthropological perspective to avoid the "lonely voyage" metaphor of cognitive construction. From this perspective, "cultural knowledge (including mathematics) is continually recreated through the coordinated actions of the members of a community" (Cobb, 1988a, p. 13). This emergent meaning becomes part of the shared world view of the participants. "It is beyond justification and has emerged as a mathematical truth for the classroom community" (p. 13). This taken-for-grantedness explains, Cobb argued, the tendency to think of mathematics as having an existence external to the knower. Mathematical concepts seem so "true" and "solid" because they have become so taken-for-granted and agreed-upon by a wide community. These meanings are so shared that they are taken for mathematical truth. They are like part of the bedrock or ultimate schemas by which we think.

Implications of learning as active construction. Whether one accepts the fundamental epistemological assumptions of the radical constructivists or simply holds the more widely accepted view of learning as a constructive process, the basic tenet that learners are active in structuring and inventing their knowledge has important implications in thinking about what it means to know and learn mathematics. The learner's existing knowledge shapes in fundamental ways what will be learned. Understanding

mathematics means having altered one's own cognitive structures or ways of thinking in powerful ways, not simply having acquired mathematical knowledge presented by others. If learning cannot be assumed to be a process of absorption or direct transfer of knowledge into the mind of the learner, then one cannot assume that what is presented through curriculum or instruction is what students will learn (Norman, 1980). Instruction can no longer be viewed as a matter of simply laying out, however carefully, the knowledge and skill to be acquired. As Resnick (1983) argued, we need to broaden our definition of instruction beyond the direct presentation of information or modeling of procedures for students to include "anything that is done in order to help someone else acquire a new capability" (p. 5), or, as Sinclair (1988) put it, to make more optimal children's efforts to "re-discover or re-invent" (p. 7) mathematical objects of thought.

But while the constructivist research agenda has highlighted the importance of attending to how children know and think about various mathematical ideas and procedures, it does not offer direct pedagogical prescriptions. In general, constructivist researchers have sought to understand the development or construction of mathematical knowledge that takes place through an individual's more or less natural interactions with the environment. It is not readily apparent, however, how this perspective should be applied to teaching and instruction. As Sinclair (1988) pointed out,

> Piagetian [i.e., constructivist] studies were not designed to discover in what kind of situations (either in or out of school) certain structures and procedures of action and thought are built up. The subtle, but powerful interaction between the societal presentation of objects which allows a great number of children nowadays to master scientific concepts only geniuses could construct in the past cries out for detailed study. (p. 9)

Just as the cognitive scientists' models of knowledge hypothesized to underlie competent mathematical performance do not tell us much about the acquisition of that knowledge, constructivist models of children's developing competence do not tell us much about how this development might be constrained or guided through powerful instructional experiences.

Understanding as Situated Cognition

Underlying the perspectives from cognitive psychology we have considered thus far is the fundamental assumption that knowledge and thinking take place within the mind of the individual.

Describing what it means to know and understand mathematics then becomes the task of describing the structures of knowledge hypothesized to exist in the individual's mind. The radical constructivists have emphasized the personal nature of the meanings that individuals construct through interaction with others and the environment, but the locus of those meanings clearly lies within the mind of the individual. A number of cognitive researchers are questioning this fundamental assumption of thinking and knowing as a process occurring within the individual's mind (Brown, Collins, & Duguid, 1989; Greeno, 1989; Resnick, 1987b), arguing instead that these events should be considered more as an interaction or relation between an individual and physical and social situations.

The move to this new perspective has been fueled, in large part, by the integration of anthropological and cultural perspectives into cognitive science. In particular, a number of studies of people dealing with quantitative problems in various out-of-school settings have revealed how solutions are constructed through interaction with the situation, rather than by applying more abstract knowledge and skills to the situation (Carraher et al., 1983; Lave, 1988; Lave, Murtaugh, & de La Rocha, 1984; Scribner, 1984). For example, Scribner (1984) found that dairy workers filling orders for cases of milk used their experience with various configurations of partially filled cases, rather than relying on school-like arithmetic procedures. As Greeno (1989) observed about their performance, "rather than assuming that there are cognitive structures and procedures that the workers applied, it seems more appropriate to say that they had acquired a capability for interacting effectively with objects in the situation" (p. 135). Rather than thinking being viewed exclusively as the manipulation of symbols and cognitive representations, it might better be considered an interaction with objects and structures of situations.

According to the situated cognition perspective, knowledge and thinking are inextricably intertwined with the physical and social situations in which they occur. Brown et al. (1989) argued that, like language, all knowledge indexes, or refers to, the world and thus derives it meaning from the situations and activity in which it is produced. Just as the meaning of a word is fundamentally dependent on the context in which it used and can never be explicated completely in a dictionary definition, all knowledge depends on the contexts in which it is used.

> A concept, for example, will continually evolve with each new occasion of use, because new situations, negotiations, and activities inevitably recast it in a new, more densely textured form. So a concept, like the meaning of a word, is always under construction. This would also appear to be true of apparently well-defined, abstract technical concepts. Even these are not wholly definable and defy categorical description; part of their meaning is always inherited from the context of use. (Brown et al., 1989, p. 33)

Emerging from this perspective as a way of thinking about conceptual knowledge is the notion that understanding could mean having a usable set of tools (Brown et al., 1989; Perkins, 1986; Stigler & Baranes, 1988). Both tools and knowledge "can only be fully understood through use, and using them entails both changing the user's view of the world and adopting the belief system of the culture in which they are used" (Brown et al., 1989, p. 33). People can acquire relatively inert knowledge of tools without actually using them, but to be able to use tools effectively requires experience using them in the ways they are ultimately to be used. For example, a person could acquire the division algorithm in the sense of being able to compute 1128 ÷ 36 but not be able to use that tool appropriately when confronted with the problem of figuring out how many buses, each holding 36 people, would be needed to transport 1128 people. Indeed, only 23% of the 13-year-old sample taking the 1982 National Assessment of Educational Progress correctly solved such a problem, with 29% of students incorrectly choosing "31 remainder 12" as the answer (Carpenter, Lindquist, Matthews, & Silver, 1983). From this perspective, knowledge for how to use tools like division appropriately is acquired through actual use of those tools in a variety of situations. For some (Brown et al., 1989; Collins, Brown, & Newman, in press), this need to acquire cognitive tools through richly embedded practice suggests that apprenticeship is an appropriate metaphor for instruction, because of its emphasis on learning in the context of authentic activity (Brown et al., 1989).

In terms of what it means to know and understand mathematics, this perspective suggests that the nature of a person's knowledge of mathematics is inextricably tied to the situations in which that knowledge was acquired. This contrasts with a long-standing tradition in education of trying to separate the desired goals or outcomes of education with its means or methods (Porter, Schmidt, Floden, & Freeman, 1986; Tyler, 1949). It also suggests that meaningful knowledge of mathematics cannot be "crystallized" and made entirely explicit. Important aspects of knowing mathematics will inherently remain

implicit and intertwined with situations in which it is used. This suggests fundamental limits to efforts to specify knowledge and understanding by explicating in as much detail as possible its components and subskills (e.g., Gagné, Briggs, & Wagner, 1988).

Thus, to learn mathematics in meaningful and useful ways, it becomes important to participate in mathematical activity--not simply to to acquire explicitly described skills and procedures of mathematics. This idea is consonant with the emphasis in the reform documents on doing mathematics rather than simply acquiring the skills and concepts that constitute the "written record" of mathematics (NCTM, 1989; National Research Council, 1989; Romberg, 1983). But what is doing mathematics? What is authentic mathematical activity? How can understanding the nature of authentic mathematical activity inform our efforts to think about what mathematical activity should be like in elementary schools? To address these questions, we turn to the discipline of mathematics.

Mathematical Perspectives on Knowing Mathematics

Defining what it means to understand a school subject has conventionally been the province of learning psychologists. As we have outlined in the first section of this paper, much of the work in this domain has focused on mathematical understanding. But we hope to go beyond psychology somewhat, and look at other ways of thinking about what it means to know mathematics. Scholars within the discipline, and those engaged in philosophical and historical analyses of what mathematics is about, have attempted both to characterize mathematical knowledge and to describe the processes whereby new mathematics comes to be accepted as true. In this section, we will look at mathematical understanding from the perspective of mathematics. Our looking to the discipline for definitions of understanding arises partly from the observation that psychology has chosen to focus on a limited set of mathematical content and partly from the inclusion in major reform documents of the assumption that students should be doing mathematics in school in order to learn mathematics (NCTM, 1989; National Research Council, 1989). This reform is seen to be necessary because school children are now learning only a very limited piece of mathematics, if they are learning any at all. Much of the mathematical activity of students in elementary school classrooms consists of practicing familiar teacher-prescribed procedures for adding, subtracting,

multiplying, and dividing numbers until they can get the right answer most of the time--both the activity and the answers being dissociated from any sense of what either the numbers or the operations mean (NCTM, 1989; Romberg & Carpenter, 1986). In contrast, reformers believe that "doing math" in the classroom ought to consist of activities like the following:

abstracting[3]	inventing	proving
applying	generalizing	conjecturing
convincing	specializing	analyzing
classifying	comparing	counting
inferring	explaining	measuring
organizing	patterning	synthesizing
representing	validating	ordering

These are the sorts of activities that are thought to characterize the work of mathematicians.

But what do these activities have to do with coming to know mathematics? What does one need to know in order to engage in such activities? How does the experience of these processes relate to understanding the body of knowledge that has accrued in our culture about mathematics? By including a section on mathematical understanding from the perspective of the discipline in this paper, we do not mean to endorse the assumption that students will learn what they need to know about mathematics by doing activities in school that are like what mathematicians do. Rather we seek to better understand the connection between mathematical practice and school learning by looking more carefully at what might be entailed in mathematical practice. Mathematical activity cannot simply be imported to the elementary school classroom without taking some care about how it is related to learning, both in terms of what a student needs to know in order to do it and in terms of what a student learns from doing it.

The reform documents have not been very clear about the distinction between what students should do in school and what they should be able to do as a result of what they learn in school. Neither have they considered the possibility that some segments of the population might come to school more disposed to learn from doing than others. It seems crucial to address these issues if we are going to be able to provide an equitable education in mathematics across the population and evaluate the effects of educational

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[3]This list is derived from such documents as the Standards (NCTM, 1989), Everybody Counts (National Research Council, 1989), California's Mathematics Model Curriculum Guide (California State Department of Education, 1987), national curriculum reform documents produced in Australia, and the British Cockcroft Report (Cockcroft, 1986).

reforms on outcomes. Do we want students to learn how to do mathematics with understanding so that we will have more mathematicians and intelligent creators of mathematical applications? Or is it that we want all students to get an appreciation of what kind of knowledge mathematics is and how truth in the discipline is warranted? Or do we want students to become intelligent users of mathematical tools in the workplace and society? These goals are not unrelated, or mutually exclusive, but distinguishing between them might help us to be better judges of what constitutes good educational practice. These issues will not be resolved in this paper, but the analysis of mathematical practice that follows may shed some light on good next questions.

What kinds of activities could be called mathematics? We would like to propose three different sorts of categories within which to think about what it means to do mathematics, and consider both theoretical and applied activity in each of these areas. We do not wish to imply that a particular bit of mathematical work or a particular mathematician falls into one category or another, however. The three activities we have chosen to analyze are interwoven in the fabric of mathematical practice.

One of the most common associations with doing mathematics is the activity of problem solving. Mathematicians use relationships among quantities and shapes to solve all kinds of problems, and problem solving is certainly something that many people would agree should be learned by students in school. We will look at the various kinds of problems that might be solved using mathematics, and what one might need to know to solve them. Another, less obvious mathematical activity is mathematizing, or building quantitative "models" of nonquantitative relationships. This activity assumes a certain view of the social and physical world, which asserts that the important elements of situations can be represented by numbers and relationships among numbers. In Western society, this view is somewhat of a given, certainly among particular segments of the population who use mathematics to formulate and solve problems and others who are consumers of their work. We don't often think of this way of thinking as something that needs to be deliberately taught. Problem solving and mathematizing are related within mathematics by the activity of problem formulating, which will be treated in both sections. The third mathematical activity we will examine is mathematical argument: How are new propositions invented in the discipline? And how is their truth established? Mathematicians make conjectures and prove or disprove

them. What kind of work goes into producing a conjecture? And what are the standards of discourse by which mathematicians decide to accept a conjecture as part of the discipline's store of knowledge?

The nature of activity in each of these domains has been regularly debated in the discipline throughout its history and continues to be a matter of considerable controversy. Understanding these debates is a monumental task for someone who is on their periphery, at the intersection between elementary education and mathematics writ large. But they cannot be ignored if we are to look to mathematical practice for guidance on curriculum and instruction. We consider three bodies of literature in the analysis of mathematical knowledge that follows: historical descriptions of the development of the discipline, examinations of contemporary mathematical practice, and philosophical treatises on the nature of mathematical knowledge.

Problem Solving as Mathematical Knowledge

Looking at mathematical understanding from the perspective of problem solving seems at first glance to be quite simple: Understanding would be knowing how to solve a mathematical problem. Assuming that the kind of solution called for is a mathematical solution (as compared with an ethical or a legal one, for example), what one would need to know to solve a problem would include heuristics or problem solving strategies, mathematical tools or conventions, and rules for justifying the appropriateness of the solution strategy. But the work of defining what kind of problem could be called a mathematical problem and what kind of solution is a mathematical solution is not quite so simple; indeed, in a recent survey of what kinds of problems contemporary mathematicians were working on, Ian Stewart (1987) called this issue an "ideological minefield." The distinction between pure and applied mathematics divides the kinds of mathematics problems considered to be worth solving as well as the practitioners who solve them. Justifications for methods are a matter of considerable controversy, particularly as a result of the current capabilities of computational technology (P. Davis, 1988; Tymoczko, 1985). When it comes to defining what is "good" mathematics, some value most those problems whose outcome matters in some material way, like the problem of figuring out how to predict the weather or the problem of determining the most efficient scheduling algorithm for airline travel, claiming that these are the kinds of problems that lead to

the creation of important new knowledge of mathematics. Others value problems whose solution adds to our accruing body of abstract knowledge about numerical and spatial relationships (Hardy, 1967; Erdos, 1988; Guillen, 1983; Gleick, 1987). An example of this latter kind of problem is determining a formula that will describe how far apart prime numbers (numbers whose only divisors are one and the number itself) are from one another. Work on prime numbers does not proceed in a way that is driven by any problem external to the development of mathematics itself. It is a matter of figuring out what the properties of abstract entities called numbers imply for patterns and relationships in the world of numbers itself.

We will look at the knowledge that one might use to solve a mathematical problem in light of both the common themes and the divergent thinking in the field about what kind of a problem is a mathematics problem, and what constitutes a mathematical solution to a problem.

Consider the following list of numbers:

1
4
9
16
25
36
49

which are the squares of the counting numbers: 1 x 1, 2 x 2, 3 x 3, and so on. Now consider the difference between each number and the one that follows it. The difference between 1 and 4 is 3, the difference between 4 and 9 is 5, the difference between 9 and 16 is 7, and so on. The differences between the squares turns out to be a list of numbers with another property: it is the list of successive odd numbers: 3, 5, 7, 9, . . . Why?

If we think of a problem in general terms as something that disturbs one's state of equilibrium because it does not fit an expected pattern (Polanyi cited in Bell, 1979), then the recognition of this unexpected regularity in the differences between successive square numbers presents us with a problem. Once the pattern is observed, there arises a curiosity about whether it will continue, and if so why. What does one need to know to solve such a problem? If the differences between the squares are represented as follows,

$$
\begin{aligned}
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16 \\
1 + 3 + 5 + 7 + 9 &= 25
\end{aligned}
$$

it is easy to see that the number of terms in each successive addition is the square root of the sum. What does one need to know in order to notice this pattern? And what does one need to know in order to appreciate that such a pattern is worthy of attention? One piece of useful knowledge to have here is that mathematics problems can often be tackled by representing them in alternative ways. This is a mathematical "heuristic" about which we will have more to say later (cf. Pólya, 1957).

The list of sums raises the question: What will the sum be in the next instance? This question was not so obvious in the earlier statement of the problem. But one must also know categories of numbers that might produce interesting results: prime, odd, and so on. What are the characteristics of prime numbers? Odd numbers? Knowing those characteristics provides some guidance in seeking to explain the pattern, but knowing that these categories exist causes us to notice the pattern in the first place. At the most basic level, one needs to know how to add a string of numbers and how to multiply one number by another to produce the pattern. These kinds of knowledge could be called mathematical tools or conventions, or more generally resources (Pea, 1987; Schoenfeld, 1985).

In the next line of the problem, there will be six terms and the sum will be 36. We can go on testing cases, obtaining more evidence that the sum of the first n odd numbers will be n^2. Mathematical argument can take us beyond such observations, however, to prove that this must be the case in every instance. Simply talking in terms of n and n^2 instead of restricting our argument to particular observations contributes something to the solution. So another kind of knowledge that is useful here is knowledge about generalization and symbolization, also referred to as abstraction (Bell, 1979; Kaput, 1987a, 1987b; Romberg, 1983). By taking a general perspective on the particular sums, looking at the elements in which an odd number of addends sums up to the square, one can see that the middle addend is also the square root of the sum. What of the elements in the list that have an even number of addends? Here, the even number that would be between the middle two terms (as 4 is between 3 and 5 in 1 + 3 + 5 + 7) is the square root of the sum. We are well on our way now to the solution of the problem of determining why the

pattern arises and whether it will continue. The introduction of a mathematical concept at this point furthers the solution: The average of the addends in each case is the square root of the sum. This auxiliary element was not part of the initial statement of the problem, but it enables us to further generalize the argument. Here we are using a knowledge of resources together with a knowledge of heuristics. In the odd cases, the average is the middle number; in the even cases it is the number between the two middle numbers. In all cases, the average of the addends multiplied by the number of addends yields the sum, and the average and the number of addends will always be the same, so the sum is a perfect square.

In the sections that follow, we will treat each of these kinds of knowledge and consider how they contribute to solving various kinds of mathematics problems.

Heuristics and mathematical induction. In the work described on the above theoretical mathematics problem, we have been carrying out a process which Pólya (1954) called mathematical induction. We discovered a pattern by observation and then proved that the pattern would continue using a mathematical argument. Mathematical induction is different from the sort of induction that leads to scientific assertions because it is supplemented by logical proof. In contrast to a simple proof, however, it reveals not only the deductions which make the conclusion inevitable, but also the experiments that led to thinking that the conclusion was plausible in the first place. Pólya calls this kind of reasoning heuristics and considers it to be the key to knowing how to solve a mathematical problem. He traces the study of heuristics back to Pappus, a Greek commentator on Euclid, and he credits Descartes, Leibnitz, and Balzano with building up a system that attempted to codify the sort of thinking underlying mathematical discovery. Pólya took up the task of describing the process of discovering solutions to mathematics problems in contemporary mathematics, partly in reaction to a trend in the discipline toward seeing the subject only in terms of its formal structure, without regard for the structure's emergence from human processes for coming to know it. He wanted to examine the elements of plausible reasoning or mathematical insight that lead to the discovery of mathematical assertions--in Pólya's terms, "heuristics endeavors to understand the process of solving problems, especially the mental operations typically useful in this process" (1957, p. 130).

Knowledge of heuristics is elusive, however. Pólya observed that it is hopeless to imagine that we will ever be able to ascertain a set of universal rules, that will yield the solution to any problem. But he does describe procedures such as "identify the unknown," "identify the conditions on the unknown," "consider a problem with a similar unknown," "introduce suitable notation," and "identify all of the data that can be used in solving the problem." And he suggests that the best way to acquire a knowledge of heuristics is to work on solving problems with someone who knows how to use them. Recent experiments with teaching problem solving to college students attest to the importance of guided practice on problems as a key to learning to use the strategies (Schoenfeld, 1985). Although it has not been a site for research on effectiveness, there is a long tradition in mathematics of national and international problem solving contests, and preparation for these contests is most often in the form of working on problem after problem with a "coach" (e.g., Loyd, 1959).

Following his own advice for teaching about heuristics, Pólya illustrates problem solving strategies by taking his readers through solutions to the sort of elementary problems with numbers as the one given above, but he asserts that these strategies are useful for solving "practical" problems as well, such as constructing a dam across a river. Recent research on teaching problem solving borrows heavily from Pólya's work, as do curriculum materials designed to help students acquire the knowledge they would need to solve mathematical problems.

To contrast with the "pure mathematics" problem considered above, we will look at an example of the sort of "practical" problem students might be given on which to practice Pólya's strategies:

> Jennifer had been begging her mother all week for some ways to earn money. First, Jennifer cleaned the garage. Then she weeded the garden. Finally, her mother agreed to pay Jennifer $3 to wash all the inside windows in the house. Jennifer worked for over two hours, completing 30 windows before her friend, Susan, came over and offered to help her. Each of them washed ten more windows and the job was done. In order to be fair, how much money should Jennifer give Susan for the windows Susan washed? (Meyer & Sallee, 1983, p. 213)[4]

The "unknown" in this problem is how much money Jennifer should give Susan. The data tell us how many windows there are, how many each girl washed, and how much money there is to be distributed.

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[4]This problem comes from a book designed to help teachers at the upper elementary level teach problem solving. It is used here as an example of a "real" problem that someone might try to solve using mathematics, not as an example of a "school" problem. The difference between school problems and real problems is a complicated one to address. See, for example, Lave & Butler, 1987; Resnick, 1987b.

"Identifying the unknown" is the first on Pólya's list of heuristics. Second is the strategy of analyzing what is known and how it might be related to the unknown but desired information. Identifying the unknown and the knowns that are related to it may seem like a fairly simple activity, but it often requires untangling relevant from irrelevant information and operating on what is given. From the knowledge structures perspective we considered earlier, this untangling depends on the problem solver having in place appropriate schemata for recognizing the relationships among quantities in the situation. The whole amount of money is stated straightforwardly in the problem, but we must do some calculations to find the total number of windows. In order to relate the known to the unknown, we must consider that the conditions on the solution are that the money should be distributed "in order to be fair." Using a mathematical concept (or what Pólya refers to as an auxiliary element and Schoenfeld calls a resource) to figure out the unknown, we could see it as a fraction of the total amount of money: Jennifer gets a fraction of the money, and she gives a fraction of the money to her friend, and the two fractions add up to the whole amount of money. How would one know to do this? By recognizing this problem as one of a type with similar unknowns, called perhaps proportion or equal ratio problems.

Here we have another kind of mathematical knowledge, which might be called knowledge of mathematical structures (Vergnaud, 1983). In these kinds of problems, what is unknown is a fraction of some quantity, and what is known is the equivalence of two fractions. The words "in order to be fair" in the context of a mathematics problem suggest that the fraction of the money each girl receives ought to be the same as the fraction of windows washed. Equivalent fractions (or a proportion) here are a tool, a resource, for shaping the problem in a way that makes it mathematically solvable. Susan washed 10 out of 50 windows, or one-fifth, so she should get one fifth of the money, or 60 cents. We can call on certain tools more readily if we know that this problem is a problem that has a multiplicative structure. (Vergnaud, 1983)

Another kind of knowledge that might go into a solution of this problem is knowing how to represent a situation using mathematical symbols. There is more to this process than simply mapping an element of the situation onto a symbol; it also requires the capacity to abstract the relevant qualities of elements and their relationships out of the situation. The equivalence between the fraction of windows washed by

Susan and the fraction of the money she should receive can be expressed in terms of a symbolic relationship:

$$\frac{10}{50} = \frac{x}{\$3.00}$$

where x represents what Susan should be paid for washing half of the remaining (20) windows.

Although we do not yet know what x represents, the problem is in some sense solved by establishing this relationship. Now we can find the value of x in two mechanical steps; first, cross-multiply to obtain:

$$50x = \$30$$

and then divide both sides of the equation by 50 to obtain:

$$x = \$.60$$

We might also simply "figure out" that replacing x with $.60 would make the ratios equal, but we would still need to prove that this insight was logically related to the conditions established in the problem.

Legitimate transformations in mathematical relationships. Behind these transformations, there is another kind of mathematical knowledge, as important in solving problems as heuristics. In order to justify the equivalence between the three equations, it is necessary to prove that it is possible to logically deduce ad = bc from the given

$$\frac{a}{b} = \frac{c}{d}$$

for any numbers a, b, c, and d except b = 0 and d = 0. This logical argument makes the truth of the equation x = $.60 an inevitable conclusion from the equivalence between the two fractions:

$$\frac{10}{50} = \frac{x}{\$3.00}$$

The problem of proving that this transformation in the relationship among the quantities in the problem (commonly called cross multiplication) is legitimate is an abstract problem whose results can be applied to any situation in which such a transformation would help to identify the unknown. It is not legitimate because of anything having to do with windows and money, but because we have precisely followed a set of mathematical laws to transform the relationships among known and unknown quantities.

There are two kinds of knowledge embedded in the use of such transformations to solve problems. One is knowing what actions are legitimate and the other is knowing why they are legitimate. Among the equations that might require such justification are

$$\frac{10}{50} = \frac{1}{5}$$

$$\frac{10}{50} \times 300 = \frac{3000}{50}$$

$$\$3.00 = 300 \text{ cents}$$

We call on knowledge of mathematical conventions, symbols, and tools as well as knowledge about how to deduce one equation from another in order to be able to use these in solving the problem. We never have to refer to the situation of windows and money and girls to justify that these equations are true. They are true by virtue of their relationship in a mathematical structure. The operations that make them true (e.g., multiply the numerator and denominator by the same number is allowed because that is multiplying by one) have no referent operations in the problem situation.

A contrasting approach: Instrumental problem solving. One way to describe the knowledge needed in the solution of the window washing problem is to say that to understand mathematics is to be able to recognize those situations to which a mathematical concept (like proportion) can be used to produce unknown information. Or, from another perspective on the above problem, we might say that understanding mathematics means being able to explain why cross-multiplication is legitimate, that is, to be able to produce a deductive argument about why the operations are legitimate given the domain of application. These ways of thinking about understanding certainly have received much attention in thinking about what mathematics students should be learning in school. But there is a third way of thinking about mathematical understanding that could be illustrated by a solution to this problem, less attended to by educators, but also derivable from considerations of how mathematics is practiced in the discipline. What one would need to know to solve the problem instrumentally is different from the knowledge outlined above, particularly in that the solution strategy is much more closely tied to the situation of the problem.

The window washing problem could be solved by estimating that the girls ought to receive something like five cents per window, passing out the money, finding out how much is left, passing out a little more, and so on, until you have passed out the whole amount. Or you might start by giving each girl ten cents per window and find out you do not have enough, and then adjust downward. The solution is justified in this approach by the actuality of arriving at a state where each girl is fairly paid for the windows she has washed. Doing this in several different kinds of problems could produce the argument that the cross multiplication technique "works" because it always produces the same result as the trial and error method. Now to one who has been trained to consider deductive proof as the only legitimate foundation for mathematical conclusions, this kind of justification may seem circuitous and circumstantial, and more importantly, solving the problem this way could be taken as an indication that mathematical knowledge is lacking rather than as an indication of a different kind of equally legitimate knowledge.

Historical perspectives on problem-solving knowledge. As the story is told by Morris Kline (1980), this instrumental approach to developing new mathematics was characteristic of Hindu mathematics in the period when algebra was being invented, and their approach to the subject was radically different from the deductive approach taken by the Greeks to the same material. Kline says of the Hindus:

> [They] were interested in and contributed to the arithmetical and computational activities of mathematics rather than to deductive pattern. . . . There is much good procedure and technical facility but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. (p. 111)

The author of the first treatise on algebra, Al-Kwarizmi (c. 830 AD), emphasized that he wished people to know mathematics that would serve their practical ends and needs "in their affairs of inheritance and legacies, in their lawsuits, in trade and commerce, in the surveying of lands and in the digging of canals" (Solomon Gandz, cited in van der Waerden, 1985, p. 15). For the Greeks, these were not mathematical pursuits; they were thought of as problems for mechanical engineers to address by practical means, and their work did not intersect that of mathematicians. This difference between the two visions of the subject led to controversies in the discipline during the Renaissance, when both Greek and Hindu mathematics were brought into play by mathematical scholars. One issue in the debate was whether negative numbers

should be allowed into mathematical analysis.[5] Pascal, a great champion of the logic of the Greeks, considered them "pure nonsense" (Kline, 1980). Argnaud, a close friend of Pascal and a renowned theologian and mathematician disallowed negative numbers from mathematics because the idea that 1/-1 = -1/1 contains a logical paradox: "How could a greater be to a smaller as a smaller is to a greater?" (Kline, 1980, p. 115.) Yet the Hindus, and the merchants who eagerly took up their efficient use of mathematics for bookkeeping, found the negative number to be a useful concept because negative numbers "worked" for balancing their accounts.

What is the difference between how "understanding mathematics" is conceived from the perspective of applying strategies derived from abstract propositions and how it is conceived if the process is seen as one of inventing a strategy that works according to the context in which it is applied? The former is certainly the more conventional view of how mathematical knowledge is used in practice to solve problems. But as the influence of Hindu algebra indicates, and a study of the development of the calculus would confirm, the latter view also has roots in mathematical disciplinary traditions and is present in current mathematical practice. The potential of mechanical computing power, which has enabled mathematicians to create tools that work to solve problems without knowing why they work, has brought about the replaying of old arguments about whether functional mathematics without a logical foundation is "real" mathematics (P. Davis, 1972; Grabiner, 1986; van der Waerden, 1985). The approach that mathematicians use for a class of problems called combinatorial optimization problems is similar to the pass out a little and reassess strategy outlined above for the window washing problem. Combinatorial optimization problems are so complex that no algorithm exists to produce a logically correct solution, even

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[5]"Without a logical way of thinking about negative numbers, without some conceptual model, [European] algebraists were unable to comprehend what it meant to add, subtract, multiply, and divide negative numbers. For that reason, negative numbers were not perceived as legitimate objects of algebraic study; their presence in certain algebraic equations was taken to have no greater significance than the existence of nonsense words in a language.

"Not until the eighteenth century did algebraists learn how standard arithmetical operations applied to negative numbers" (Guillen, 1983, p. 63).

with the most powerful computer[6] (cf. Stewart, 1987). Devlin (1988) considers these kinds of problems to be "an exemplary blend of the pure and the abstract" and an indication that mathematics is entering a new Golden Age in which the distinctions between materially useful solutions and theoretical developments in the discipline will be less well defined.

Solving problems like these ties mathematics to the physical and social world in ways that make more theoretical members of the discipline uncomfortable. However, like Devlin, Kline (1980) believes that these ties are what drives mathematical development:

> How did mathematicians know where to head, and in view of their tradition of logical proof, how could they have dared merely to apply rules and assert the reliability of their conclusions? There is no question that solving physical problems supplied the goal. . . . The physical meaning of the mathematics also guided the mathematical steps and often supplied partial arguments to fill in non-mathematical steps. (p. 168)

Kline's argument does not explain, however, how it is that the development of mathematics simply out of an interest in generating new abstract knowledge later comes to be useful. This is the case with many inventions in the field of analysis, where new kinds of numbers have been invented purely to extend the range of a theoretical notion, for example, complex numbers which take the form of multiples of $\sqrt{-1}$ (Guillen, 1983). At the same time, Stewart (1987) reminds us that if cryptographers did not have certain theorems of prime numbers, like Fermat's, available for use, "They would have played around on a computer, found it as an empirical rule, and no doubt used it without worrying too hard about a proof. If a proof had seemed necessary, it would have been sought and found" (p. 231).

Here we return to Pólya's notions about mathematical induction as the essence of mathematical problem solving, and also to one of the deepest controversies in current mathematical practice; that is, whether a mathematical assertion that rests on computational power, without concurrently being supported by a deductive argument, can be considered a mathematical truth (Davis, 1972; Devlin, 1988; Tymoscko, 1985). And in terms of the sort of mathematical knowledge or understanding that schools

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[6]An example of this sort of problem is the traveling salesperson problem in which the challenge is to find the shortest route among a given list of cities where each city is to be visited exactly once. For more than a few cities, the number of combinations is so large that each cannot be evaluated to find the best. Instead, mathematicians work on assessing how much better or worse one alternative is than another, which they can do by using computers to simulate different routes. Their results are of considerable interest to urban planners concerned to design efficient public transportation and to those whose work it is to decide what would be the most efficient location for commercial airline hubs.

should produce, we must now ask, "Is it enough to know that it works, without knowing why it works?" And what sort of argument is acceptable in order to assert about a piece of mathematics "that it works"? These questions take us right to the edge of thinking in the discipline about what mathematics needs to be known in order to solve mathematical problems, but these questions are not very far from the sorts of questions a teacher might wonder about when charged with producing mathematical understanding in her students. These issues will be taken up from another perspective later in this section as we consider what it means to understand mathematics as mathematical argument.

Mathematizing, or Reducing a Situation to Its Quantitative Relationships

Mathematizing, or mathematical modeling, is the activity of representing relationships within a complex situation in such a way as to make it possible to put them into quantitative relationships with one another, and thereby find out new information about the situation by solving numerical equations. For example, faced with the problem of deciding which 200 out of 1000 highly qualified applicants should be accepted into a college's freshman class, most admissions offices make the decision by assigning numbers to the candidates qualifications, adding up the numbers, and taking those with the highest composite scores. Or in the realm of weather predicting, faced with the ominous consequences of a global warming trend, scientists quantify those activities which seems to be causing the trend, and use relationships among these quantities to recommend emissions controls and resource development policies. Mathematical modeling requires having the knowledge to be able to decide what does not matter in making the model, and the disposition to accept the answers that the numerical procedure provides as a valid solution to the problem. It is this way of approaching problems that has enabled us to make such wide use of computers as problem solving tools (Dreyfus & Dreyfus, 1986; Wiezenbaum, 1976). The computer does not mathematize, but once a mathematical model is constructed for a situation, it can quickly "crunch" all of the numbers that are fed into it and produce a solution to a complex problem.

Mathematical understanding, from the perspective of mathematical modeling, is knowing how to find patterns and relationships in, or impose them on, nonmathematical phenomena. It is knowing how to distill the mathematical essence out of a messy situation, and once the model is established, knowing how to

define the conditions under which it is useful and appropriate. It is knowing how to identify those situations in which it is productive and appropriate to apply mathematics.

Distilling the mathematical essence. One of the earliest examples of formal mathematization in Western history is Eratosthenes' attempt to determine the size of the earth[7] in about the third century B.C. (Pólya, 1963/1977). By taking as a given that the earth was a sphere, he reduced the problem of finding size to the problem of finding circumference; his calculations assumed that the longest straight path around the earth would be a circle, and thus its length could be related to the mathematical idea that a circle contains 360°. His method was to establish certain quantitative relationships (between the angle of the sun's rays at given points and the distance between those points) and to reason logically from those relationships to others. He needed to adjust many realities to his formulas (like assuming that the rays of the sun at one point were sensibly parallel to those striking a nearby point because the sun is so far away), reasoning about what would affect the results of the calculation and how. His methods are very similar to those used today in creating maps, and there is evidence to suggest that such methods were also used by the Chinese circa 720 A.D. and by the Muslims circa 820 A.D. (Morrison & Morrison, 1987). By viewing reality through the lens of a theoretical mathematical structure, Eratosthenes could figure out the radius of the earth without measuring. In Pólya's words, "A mere shadow and an idea is the substance that made the pigmy a giant who spanned the earth" (1963/1977, p. 14). It was Eratosthenes' mathematical modeling of the problem that enabled him to find a solution.

As can be seen from this example, mathematization has two parts. First, it requires extracting from their context those elements of a situation deemed to be relevant and placing them into a quantitative relationship to one another using some mathematical construct. Mathematical transformations of this relationship are then used to establish other quantitative relationships, and the results are applied back to the situation. Eratosthenes began with the sun's inclination to the vertical at Alexandria (7°12') compared

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[7]Counting, or establishing number, is itself an act of mathematization in that it involves a disposition to attend to an abstract property of objects rather than to their particular differences. Eratosthenes' work was continuous with this way of thinking about the world, but constituted a more conscious attempt to use mathematical structures.

with its angle at Syrene, measured the distance between Syrene and Alexandria, and used the mathematical relationship

$$\frac{7^\circ 12'}{360^\circ} = \frac{1}{50} \text{ of a complete revolution of the earth}$$

to relate the Syrene-Alexander distance to the distance around the earth. Because it is mathematically true that there are 360° in a circle, and because the ratio of degrees in one segment of the sphere to the whole is equal to the ratio between distance in the same segment on the surface of the sphere, Eratosthenes could conclude with certainty about a phenomenon he could not observe. Dividing a circle into degrees and relating numbers in proportions are mathematical constructs he used to do this work. The process does not exactly predict the circumference of the earth, rather it gives you a pretty good sense of what you would find if you ever decided to measure it.

Being disposed to see mathematics as powerful. Descartes made his place in philosophy by expressing the hope that all physical and human relationships could be represented in the sort of clear and direct mathematical terms that Eratosthenes applied to understanding the size of the earth. He assumed that if this could be done, all matters from morals to mechanics could be decided with certainty. His program was played out during the Age of Reason, in which great strides were made in the physical sciences by quantifying the relationships between physical properties of matter. Beginning with the establishment of the heliocentric theory of planetary motion, mathematics became the basis for challenging traditional knowledge and proving that things are not always what they appear to be. With its strong ties to logic, mathematics was considered to be an unquestionably true basis on which to found knowledge in other domains. The incredible, predictive power of mathematics--for example, to establish, the certain existence of the planet Jupiter before it was ever viewed by a telescope, or to point to the existence of as yet undistilled chemical substances because of their position on a numerical scale of atomic weights--gave both scientists and the public the belief that mathematics must be all powerful (cf. Judson, 1987).

Mathematics was thought by Descartes to be a general science "beyond subject matter," and therefore to be the foundation of all knowledge:

> The long chains of simple and easy reasonings by means of which geometers are accustomed to reach the conclusions of their most difficult demonstrations had led me to imagine that all things to the knowledge of which man is competent are mutually connected in the same way. (cited in Kline, 1985, p. 90)

Descartes argued that since man can comprehend mathematics, the world must be organized along mathematical lines, and even God's existence could be proven by mathematical methods. Mathematics appealed to Descartes because it was less mystical, less metaphysical, and less theological than the routes to knowledge followed by his medieval and Renaissance predecessors. Mathematics was a way of knowing that did not depend on acquiescence to authority: Man could look inside himself and decide, by logical reasoning, what was true.

In nineteenth-century France, what Davis and Hersh (1987) call "Descartes Dream" led to the creation of the social sciences. Auguste Comte studied Lagrange's attempt to reduce all of mechanics to mathematics, and reasoned that "if physics was built on mathematics, so was chemistry built on physics, biology on chemistry, psychology on biology, and finally his own new creation, sociology (the term is his) would be built on psychology" (Grabiner, 1988, p. 225). This notion that mathematics could be the foundation on which to build certainty about social phenomena is echoed in contemporary attempts to use social science for social problem solving;[8] theoreticians of social policy aim for what they call programmatic rationality--"to achieve substantive goals through instrumental action programs that can be proven logically or empirically, to achieve those goals" (Gans cited in Lindblom & Cohen, 1979, p. 31, emphasis added). The belief underlying this quest is that the relationship between mathematics and truth in the solution of scientific and social problems will free us from tradition, prejudice, and the preponderance of power (Kaplan cited in Lindblom & Cohen, 1979). The assumption that expressing problems in terms of mathematical relations between quantities will remove their solution from the realm of human judgment and folly pervades every aspect of our lives. We hope to use mathematics to remove gender biases from college admissions exams (Berger, 1989), for example, and to rectify the inequities in charges that physicians make for different kinds of services (Andrews, 1989).

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[8]See, for example, Tufte (1970). For a critique of the quantitative approach to human problem solving, see Braybrooke and Lindblom (1963) and Wiezenbaum (1976).

Teaching students to make mathematical models obviously has enormous potential for improving both their ability to conceptualize problems and their ability to appreciate the attempts that are made to solve physical and social problems using mathematics. But we might also want to consider whether there might also be pitfalls in engaging students in mathematizing because of the controversy that arises within the discipline about this way of doing mathematics. Davis and Hersh (1987) observe that there are common intellectual dispositions among those who are "doing mathematics" for a living, and that these dispositions are both powerful and problematic. "Confronted with a fuzzy universe [the mathematical mind] tries to find precise statements about that which is chaotic or random" (p. 124). It is this disposition that enabled eighteenth-century mathematicians to find the order in the solar system that supports today's explorations of the planets, but this is also the sort of disposition that "dehumanizes" human phenomenon to produce statistics:

> Statistics (as opposed to mere data collection) begins when one agrees to form averages. Bill weighs 168 pounds, John weighs 190 pounds, and Bobby weighs 161 pounds. Their average weight is 173 pounds. This last statement is a composition of the first three. There is a loss of meaning in passing from the first three numbers to the fourth. There is, of course, a gain in the recognition of the empirical fact of the stability of averages. It may be that one of the reasons why probability and statistics did not take off until the 17th century was precisely the refusal of people to suffer the loss of the sense of the individual. . . .
>
> Whenever anyone writes down an equation that explicitly or implicitly alludes to an individual or a group of individuals, whether this be in economics, sociology, psychology, medicine, politics, demography, or military affairs, the possibility of dehumanization exists. Whenever we use computerization to proceed from formulas and algorithms to policy and to actions affecting humans, we stand open to good and to evil on a massive scale. What is not often pointed out is that this dehumanization is intrinsic to the fundamental intellectual processes that are inherent in mathematics. (Davis and Hersh, 1987, p. 282-283)

Davis (1988) has advocated the idea that mathematics education might take up the task of teaching students to think more critically about the application of mathematical methods to the solution of problems. He points out that the application of mathematics to the way we understand and organize the world is a social contract rather than the discovery of innate characteristics of situations. The idea of averages, for example, that underlies so much work in the social sciences, is an agreement to disregard certain elements of individual identity, which are not always appropriate to disregard.

Deciding when mathematical modeling is appropriate. The allocation of donated organs to patients in need of a transplant is an example of the sort of problem that might be solved differently using mathematization than it would be solved from a nonmathematical perspective. The quantification of life

expectancy, probability of recovery, family situation, and so on, does not quite capture all that needs to be considered in who gets the next available heart transplant. On the basis of several such examples of "mathematical modeling," the mathematician Douglas Hofstadter (1986) concurs with Davis and Hersh's worry about the capacity of numbers to dehumanize the way we think about situations in which the problems arise, creating a kind of "number numbness" to mask the complexity of problem solutions.

For a consideration of what learning to do this kind of thinking might mean at the elementary school level, we will look again at the problem of trying to figure out how much each girl should be paid for washing windows posed above on page 50. To "mathematize" this problem situation, one does things like count the total number of windows and figure out how to connect the money that each girl is paid with the number of windows she has washed. The "context," that is, that Janet had been "begging" her mother for money, that she had already done several other jobs, and so on is to be disregarded as mathematically irrelevant. In mathematical terms, a fair distribution of the money should be proportionally related to the number of windows washed. We can apply the operations of addition and multiplication to the relationship between the windows and the money in this problem because we assume that for the purposes of our solution, every window is equivalent to every other window; their relative size or dirtiness, and when during the job they were washed do not matter. In order to be able to use mathematical tools to arrive at a solution to the problem, we must attend only to how many windows there are and how many each girl washed. We know that the last 20 windows to be washed were washed by 2 people, each washing 10 windows. Again, using the operation of division assumes that what is divided gets divided into equal groups. So since the friend washed 10 windows, she gets 60¢. The daughter washed 30 and then 10 more, 40 altogether, so she gets $2.40. And that totals up neatly to the $3.00, which the mother offered, confirming the appropriateness of our calculations.

But there are other resolutions to the problem which would not be obtained by the process of mathematization, and might be equally likely to be proposed by someone who did not immediately see

this as a mathematical problem.[9] Maybe the friend should not get any of the money because it was promised in a contract between the mother and daughter, in which she had no part. Maybe the mother should not give the daughter any money after all because she and her friend were having so much fun that the window washing could hardly be considered work. Maybe the friend should pay the daughter for the privilege of participating in the window washing activity (à la Tom Sawyer whitewashing the fence). Maybe they should split the money evenly to be fair, since the friend was not really given a chance to wash half the windows from the beginning (à la the parable in the Bible about the fieldworkers who arrived at different times of the day). Maybe the mother should pay the friend something over and above the $3.00 she was planning to pay her daughter because maybe the daughter made the agreement because she needed three dollars for something particular that she wanted to buy. Maybe you can't even figure out what is fair, because you don't have any information about how long the job took all together and you don't know whether some windows were bigger than others, or harder to reach, or dirtier. These considerations serve to emphasize the character of the kind of thinking that is involved in "doing mathematics" on a problem situation like this. Mathematics focuses on quantitative relationships, and by so focusing is able to generate new and useful information, but only by leaving aside all of these other considerations. Teaching students to mathematize could be done in a way that treats all of these alternatives as a distraction from "really solving" the problem, or it could consider the strengths and weaknesses of various approaches.

The problem in taking the mathematical activity of modeling or mathematizing as a route to thinking about what should occur in elementary curriculum and instruction is getting some distance on the assumptions that are made in our culture about the objectivity of mathematical models and the moral dispassion that is associated with the solutions produced by them. Mathematics is a powerful tool for constructing and extending relationships among quantities, but its power is not always a positive force. In

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[9]The following alternatives were indeed proposed when Lampert set a group of elementary school teachers thinking about the window washing problem during her Teacher Study Group on Mathematics (Lampert, 1988b).

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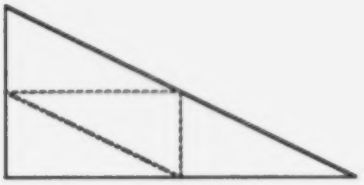
the endeavor to get learners to do the mathematics of abstracting, modeling, ordering, and classifying, we should not lose sight of the perspective on the world that these activities entail.

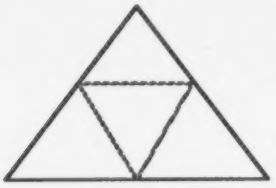
Mathematical Argument, or What Is True In Mathematics and Why?

In addition to solving problems and creating mathematical models, mathematicians "prove things." The distinctive content of mathematics is relationships among quantities and among shapes, and the association of quantity with qualities of shape. In mathematical argument or discourse, assertions are made about these relationships, and those assertions are proven to be true or false using logical deduction from agreed-upon assumptions. This is a simple description of a highly contested terrain, but it allows us to get into the issue of what mathematicians do when they are working to generate new mathematical knowledge.

Conventionally, the Elements of Euclid have been considered to represent the ideal of mathematical argument. In Euclid's geometry, numerous theorems which relate characteristics of geometrical forms are generated by logical deduction from a few axioms and definitions. The theorems are assertions about relationships among shapes which can be "proven" to be true using the rules of logic and axioms and definitions and other previously proven theorems. The geometrical forms about which theorems are asserted are abstractions; that is, they are not actual triangles or squares or circles drawn on paper or otherwise present to the senses, but they are ideas. Working on such ideas gives mathematicians the capacity to make statements about what is true of "all triangles"; these statements are shown to be true based on chains of logical reasoning from what we know about all triangles already. By definition, a triangle is any closed figure in a plane with three straight sides; everything else we "know" about triangles proceeds from accepting that definition and a few "self-evident" assumptions (axioms) about figures in the plane. We can know for certain within this system, for example, that the triangle that is formed by joining the midpoints of the sides of any triangle will have three angles that are the same size as the angles in the original triangle, and that its sides will be exactly half as long as the sides of the original triangle. This is true whether the triangles look like this:



or like this:



or like any similar pair of triangles you can imagine. Proving that this assertion is true does not involve drawing and measuring lots of triangles, however. It involves making deductions from what we already know is true about angles, lines, and triangles, in the abstract. For example, we know that the line joining the midpoints of two sides is always parallel to the third side, that certain angles that are formed when a line intersects two parallel lines are equal, and so on. One way of thinking about the "goodness" of a mathematical system is to look at whether it can produce many theorems out of only a few axioms and definitions. And Euclidean geometry is often used as an example of the power of mathematical deduction because it does just that.[10] Using Euclidean geometry as the ideal of mathematical argument, we could conclude that mathematical understanding means being able to produce a deductive proof for any assertion.

What kind of knowledge is entailed in proving? There is a problem with thinking about Euclidean geometry as the prototype for mathematical thinking because it emphasizes the process of proving conjectures without saying much about where the conjectures come from in the first place or what conjectures might have to do with understanding before they are proven. Once a conjecture in geometry or any other branch of mathematics is stated, its truth often seems obvious, especially to the person who

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[10]The status of Euclidean geometry as a body of mathematical truth has changed since the invention of several non-Euclidean geometries in the nineteenth century. Before that development, the axioms and definitions were considered to be true abstractions from real world figures. In current thinking, they are considered as a set of agreed upon assumptions that provide a starting point for deducing other theorems. This does not change the status of this body of knowledge as an example of an axiomatic system, however.

stated it. It is not clear, from this perspective, what the proof of the conjecture would add in the way of understanding. This is true at the highest levels of mathematical creation, as well as at more elementary levels of mathematical argument. In mathematical practice, the quality of a conjecture has a great deal to do with whether it is formulated in such a way as to make the forthcoming proof seem uneventful:

> When a mathematician asks himself why some result should hold, the answer he seeks is some intuitive understanding. In fact, a rigorous proof means nothing to him if the result doesn't make sense intuitively. If it doesn't, he will examine the proof very critically. If the proof seems right, he will then try hard to see what is wrong with his intuition. . . . Poincare said, 'When a somewhat long argument leads us to a simple and striking result, we are not satisfied until we have shown that we could have foreseen, if not the entire result, at least its principle features!" (Kline, 1980, p. 312)

A conjecture is more than a guess, and must be judged according to the quality of the evidence that supports it, even when strict logical evidence that it is true has not yet been produced. In Poincare's reflections on his own work as a mathematician, insight or intuition seems at least as important to securing mathematical understanding as logical proof (Hadamard, 1945). This phenomenon in mathematical practice is a serious challenge to the primacy of the deductive process for supporting an argument represented by the Euclidean ideal.

Knowing how to make conjectures. Can we say that a learner understands a bit of mathematics about which he or she has made a conjecture, even if no logical proof of the conjecture can be produced? Both conjectures and theorems require what is commonly called mathematical reasoning but the reasoning that leads to a theorem is much more transparent than that which leads to a conjecture, and thus much easier for the listener to evaluate. Pólya (1957) makes the distinction as follows:

> We secure our mathematical knowledge by demonstrative reasoning, but we support our conjectures by plausible reasoning. . . . The difference between the two kinds of reasoning is great and manifold. Demonstrative reasoning is safe, beyond controversy, and final. Plausible reasoning is hazardous, controversial, and provisional. . . . Demonstrative reasoning has rigid standards, codified and clarified by logic (formal or demonstrative logic), which is the theory of demonstrative reasoning. The standards of plausible reasoning are fluid, and there is no theory of such reasoning that could be compared to demonstrative logic in clarity or would command comparable consensus. (p. v, emphasis added)

Here Pólya is saying that deductive arguments follow agreed-upon rules, and so their validity can be evaluated by others who are familiar with these classical standards. No such consensus operates, he claims, about what constitutes a legitimate argument in support of the plausibility of a conjecture.

The simplest way to make the distinction between justifying conjectures and justifying theorems is to assert that conjectures are the result of induction; that is, they are the result of observing patterns in a phenomenon, and with good reason, asserting that the pattern will continue in a way that leads to some general truth. Of course the truth of the conclusions, even those based on demonstrative reasoning, depends on the assumptions that are made, and many mathematical "truths" have required revision when the assumptions on which they were based were challenged. Assumptions can be challenged with one counterexample; it is not the universally accepted logical process that is called into question (Lakatos, 1976). And there are no hard and fast rules for how to find the counterexample that will do the trick. So in actual mathematical practice, there is less of a distinction between induction and deduction, or between intuition and logic, than scholastic definitions would lead us to believe. Plausible reasoning produces counterexamples, counterexamples require the revision of assumptions, and the process results in the refinement of the demonstrative argument. This raises important questions for those who would have students making conjectures and producing counterexamples to challenge the assertions of other students at the level of elementary mathematics. How are we to assess what the student understands if there are no standards in the discipline with which to judge the adequacy of "plausible reasoning"? How does one learn to have the mathematical insight that leads to good conjectures or powerful counterexamples? Is it simply a matter of being socialized to pay attention to one's own sense-making capacities? Or is mathematical insight a "talent" with which only a few are endowed?

Freudenthal (1978) writes at length about the student who answers "I see it so" (or in Dutch, "Ik zie het zo") when asked how they have solved a mathematical problem. He, as well as Wheeler (cited in Bell, 1979) and Bell (1979), have considered versions of the problems stated above and the implications of their possible answers for mathematics education. Each of these writers wishes to acknowledge conjecture as a legitimate form of mathematical thinking because it is considered with respect in the work of the discipline. They all attend to the capacity that we have to imagine a mathematical generalization from one significant example. Freudenthal (1978) calls such examples paradigms, using this term in a somewhat different way than it is commonly used. In his studies of students' attempts to convince their peers of a mathematical assertion, Balacheff (1987) considers the various ways students use examples to

make an argument. He distinguishes the crucial experiment from the naive experiment: The former relies on choosing an example that does not facilitate the truth of the statement. Even more powerful is the students' use of the generic example, which relies on knowing enough to choose an example that cannot in any way be construed as a special case, a "good representative" of the class of objects to which the assertion is to apply. Balacheff considers these forms of argument to be developments toward a mathematical way of thinking about what constitutes appropriate evidence. There are two related phenomena here which raise questions about how mathematical understanding can be acquired and evaluated: One is the idiosyncratic nature of insight, and the other is the attempt to test the validity of one's insights by trying to argue their plausibility in a community of peers. These issues in mathematics education are intricately related to issues of how new mathematical knowledge is generated in the discipline, not because education is about preparing mathematicians, but because these issues go directly to the heart of what it means to know mathematics and how that knowledge is acquired.

The relationship between conjecture and proof. In Proofs and Refutations, Imre Lakatos (1976) portrays historical debates within mathematics about what the "proof" of a theorem represents by constructing a conversation among a group of students: fictional characters who voice the disagreements among mathematicians through the last several centuries, often using the mathematicians' own words. Lakatos's argument, which comes through in the person of the teacher, is that mathematics develops as a process of "conscious guessing" about relationships among quantities and shapes, with proof following a zig-zag path starting from conjectures and moving to the examination of premises through the use of counterexamples, or refutations. This activity of doing mathematics is different from what is recorded once it is done: "Naive conjecture and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end product" (Lakatos, 1976, p. 42). The product of mathematical activity might be justified with a deductive proof, but the product does not represent the process of coming to know. Nor is knowing final or certain, even with a proof, for the assumptions on which the proof is based-- the axioms accepted as self-evidently true by the people who work in that branch of the discipline--continue to be open to reexamination in the mathematical community of discourse.

Mathematics has grown and changed over time, in Lakatos's view, not because the conclusions that are derived from axioms are the result of faulty logic, but because the axioms and definitions from which the logical argument begins are themselves open to revision as they are examined in the community of discourse. The need for revisions does not become obvious, however, until one engages in the process of proof, and discovers the shortcomings of one's assumptions. The insufficiency of the original assumptions comes to be recognized as one tries to pursue their logical consequences rather than before the fact: Refutations of the conclusions, often in the form of counterexamples, suggest revisions to the assumptions. Lakatos demonstrated that this zig-zag between revising conclusions and revising assumptions in the process of coming to know occurred both in the work of individual mathematicians as they exposed their work to their colleagues, and over time as conclusions that had been unquestioned in the past were reconsidered.

From the standpoint of the person doing mathematics, making a conjecture (or what Lakatos calls a conscious guess) is taking a risk because the process of mathematical argument is social; conjecturing requires the admission that one's assumptions are open to revision, that one's insights may have been limited, that one's conclusions may have been inappropriate. While possibly garnering recognition for inventiveness, letting other interested persons in on one's conjectures increases personal vulnerability. Lakatos asserted that in order to do mathematics, a scholar needed to have the courage to make guesses about what might be true in a system of mathematical relationships, and then have the modesty to examine--and let others examine--the assumptions behind those assertions. Courage and modesty are appropriate to participation in mathematical activity because truth remains tentative, even as the proof of a conjecture evolves. It is often the case that a conjecture is asserted by one individual and proven by another. In the research literature, mathematicians' names are associated both with conjectures and with proofs.

Pólya (1954) also thought courage and modesty to be essential to the activity of acquiring new mathematical knowledge. He asserted that the doer of mathematics must assume "the inductive attitude," and be willing to question both observations and generalizations, playing them off of one another in a form similar to what Lakatos called the zig-zag path from conjecture to proof and back to axioms. Pólya's

emphasis on the role of induction in mathematics is intended to complement the more formalized view that mathematics is about proving theorems. Like Lakatos, he is concerned with helping students and other nonmathematicians to understand where the theorems come from in the first place, and what we really know when we have proved one.

What mathematicians do when they are creating mathematics is a relatively new focus for philosophers of mathematics. Until recently, studies of mathematical argument focused on the nature of the connection between the truth of mathematical assertions and their logical foundations, with scant attention to how the assertions came into being. What has brought the question of mathematical practice to the forefront in contemporary work are some rather profound failures on the part of the logicists and the formalists to secure the foundations of mathematical certainty by attaching it to a deductive structure (Tymoczko, 1985). The more philosophers pressed for consistency and logic in the foundations of mathematics, the more inconsistency and paradox they discovered (Kramer, 1970). And so, in the late twentieth century, the fallibilist view of Lakatos and Pólya, with their attention to the process of generating and revising mathematical conjectures in social discourse, has gained attention.

Lakatos (1976) calls mathematics quasiempirical because of the way it depends on counterexamples to refine assertions, and Pólya (1954) asserts that doing mathematics requires "an inductive attitude" to discern patterns in numerical and spatial relationships. But the discourse of mathematics is distinguished from scientific discourse because mathematics does not depend for its verity on physical evidence. Whether one is considering a conjecture or a theorem, knowing it mathematically depends on whether the assertion "makes sense" in an informal, intuitive way (for conjectures) or in a formal, logical way (for theorems). Mathematicians can do their work--the work of inventing and justifying mathematical assertions, without reference to empirical data, although they do use "images" drawn from experience in the world to reason about ideas (Gleick, 1987; Noddings, 1985). A common observation in several recently published popular descriptions of mathematicians at work also found in some classical historical accounts, is that mathematicians do not need a laboratory; they have ideas while they sit or stand at their desks or at the kitchen table, or they figure things out while getting on the street car or sitting around waiting for a meal in a restaurant. In those settings, they can test the validity of their ideas as well

(e.g., Cole, 1987; Gleick, 1987; Hadamard, 1945; Hoffman, 1987). This habit of working is worth noting because it is directly related to the issue of verification of beliefs. The relation between justification and certainty in mathematics is mysteriously ironic, and has puzzled thinkers continuously over the centuries: How can we be especially certain that assertions of a mathematical sort are true when there is no physical evidence? In common sense terms, if a scientist claims that iron rusts in the presence of oxides, anyone can apply oxides to a piece of iron to observe whether the scientist is telling the truth. But what of conjectures that are put forth with considerable certainty and even used in the solution of problems when no proof has yet been produced? These questions are of interest as we think about students "conjecturing" at the elementary level. What does being able to conjecture suggest they understand? What do they need to understand to be able to do it?

Consider the following conjecture: The number of even numbers is equal to the number of numbers. Why is it true? Whenever you double a number, you get an even number because an even number is defined to be a number that has 2 as one of its factors, and you always get such a number when you double a given number. So every number has a partner that is an even number. It is impossible to find a case that contradicts the idea that there are as many even numbers as there are numbers. While the scientist follows canons of experimentation that set standards for relating sample size to conclusions, the mathematician does not even need to test one case. It might be said, from this comparison, that the scientists' process of verification, or even the similar, more commonsense process by which we can check out whether the person who says it is raining outside is telling the truth--is a public process, while the mathematician's justification is private. In ordinary terms, it occurs inside the knower's head. But the mathematician's justification to him or her self is not enough to establish an insight as part of the discipline's body of knowledge. An argument must be constructed that convinces other interested parties of the truth of the insight, and at this point, private knowledge becomes public in mathematics (Tymoscko, 1985).

Mathematical argument as a social phenomenon. Davis and Hersh (1987) point out that the mathematician's public arguments do not match the ideal of a deductive logical argument, however. They assume that the interested others who are reading or hearing the argument share the concerns and

values that guided its development, allowing work to proceed within a community of scholars without step-by-step deductive proofs.

> In the real world of mathematics, a mathematical paper does two things. It testifies that the author has convinced himself and his friends that certain "results" are true, and it presents a part of the evidence on which this conviction is based.
>
> It presents part, not all, because certain routine calculations are deemed unworthy of print. The reader is expected to produce them for himself. More important, certain "heuristic" reasonings, including perhaps the motivation which led in the first place to undertaking the investigation, are deemed "inessential" or "irrelevant" for purposes of publication. Knowing this unstated background motivation is what it takes to be a qualified reader of the article.
>
> How does one acquire this background? Almost always, it is by word of mouth from some other member of the intended audience, some other person already initiated in the particular area of research in question. . . . Mathematical argument is addressed to a human audience, which possesses the background knowledge enabling it to understand the intentions of the speaker or the author. (Davis & Hersh, 1987, p. 66)

Considering the social character of mathematical knowledge--in light of the question of what constitutes "proof" in mathematical practice--raises some important questions to attend to as we think about what it might mean for students to "do" mathematical argument at the elementary level. If we give up the idea that mathematics is about ultimate truths and see it instead as a human construction, it becomes much harder to separate disciplinary conventions from logical necessities. If we put emphasis on students' doing mathematics, and interpret that to mean constructing mathematical arguments in the classroom community of discourse, we need also to ask how this discourse ought to be related to more advanced work in the field and to the the heritage of mathematical concepts that have been constructed and used over time.

But there is an even deeper issue here, and a fundamental irony. If, as Davis and Hersh (1987) point out, there are agreed-upon and unexamined assumptions within mathematical practice, not only about terms and symbols, but about legitimate heuristics and appropriate motivations for undertaking a problem in the first place, then what are we to make of the idea that mathematics is powerful precisely because it enables us to know things without reference to authority? If learning mathematics means becoming indoctrinated to the rules of discourse, how can learning mathematics also mean being educated to pay attention to one's own capacity for sense making? Within communities of working mathematicians, as within all such working communities, there are a host of agreed-upon assumptions: assumptions whose legitimacy is taken for granted so that they can get on with the conversation. Working mathematicians may

not stop to convince themselves or anyone else that these assumptions are legitimate: Being a member of the discourse simply implies being willing to play by the same rules that everyone else is playing by. This is an important issue for educators to consider because elementary schooling is not about initiating students into the society of working mathematicians, although it might serve the purpose of giving them some basis for choosing whether this is the society they want to join. Mathematical arguments can not be imported whole into settings where the audience does not already understand something of what they are about. And yet we know that it is possible for elementary school children to make mathematical assertions and to produce mathematically legitimate arguments for their validity. What we need to work out is how one learns to do that, how such learning might be documented and evaluated, and whether this is what ought to constitute education in doing mathematics.

Focus on Classroom Practice

Thus far we have considered views of what it might mean to know and understand mathematics from the perspectives of cognitive psychology and mathematics. At this point we turn to perspectives emerging from a focus on the elementary school classroom. We begin by considering the assumptions, often implicit, of researchers studying classroom teaching of mathematics in the effective teaching, or process-product, approach. This research program is important in part because it builds on assumptions about the the nature of mathematical knowledge and goals for schooling that are implicit in existing practice, and in part because it has had such a powerful influence on educational policy in recent years. As a contrast to process-product research, we then consider three recent cases of researchers studying mathematics teaching and learning in classrooms by attempting to change some of the fundamental assumptions about knowing and learning mathematics.

Research on Effective Teaching

Research on teaching blossomed in the early 1970s with the adoption of the process-product approach, whose proponents sought to discover stable relationships between teacher behaviors--process variables--and student achievement and attitude outcomes--product variables (Brophy & Good, 1986; Gage, 1978). Process-product researchers were motivated by dissatisfaction with laboratory-based

theories of learning for informing teaching practice (Gage, 1972), and with research that "avoided looking at the actual processes of teaching in the classroom" (Dunkin & Biddle, 1974, p. 13). By focusing on what teachers do in the classroom, these researchers hoped to find behaviors and strategies for teaching that worked--that could be deployed by teachers to become more effective, with effectiveness for the most part defined as student achievement measured by standardized tests. Their quest was for a scientific basis for teaching, with science defined as the search for relationships between variables (Gage, 1978), and the variables being those derived from classroom teaching, not laboratory studies of learning.

The process-product research strategy was successful in producing a consistent picture of teaching that was effective in producing achievement gains: teaching that was highly structured and directed, involved explicit explanation and modeling by the teacher, and kept students highly engaged with academic content (Brophy & Good, 1986; Rosenshine & Stevens, 1986; Shulman, 1986). But process-product research has been criticized on a number of counts, including its lack of attention to the subject-matter content being taught (Romberg & Carpenter, 1986; Shulman, 1986), its focus on observable behavior and resulting disregard for the cognitive activities of teachers and students (Shulman, 1986), and its inherent conservatism arising from the study of existing practice (Romberg & Carpenter, 1986). Our goal here is not to provide another critique of process-product research, but to examine its assumptions about the nature of knowledge and learning. Because this research program has had such a significant impact on educational policy, it is important to examine these assumptions, both to understand their relationships to the various perspectives we have already discussed and to inform our emerging conceptions of what knowing mathematics might mean in classrooms.

Separation of teaching and content. Because process-product researchers were seeking generalizable findings about teaching, they generally made the assumption that teaching could be usefully separated from the content being taught. This assumption was consistent with the search for general laws of learning in process-product research's parent discipline of psychology. Deciding upon the content of instruction was considered to be heavily values-based and to be the proper focus of curriculum experts and public debate, not of research on teaching. The goal of research on effective teaching was to answer the question, Given particular goals of instruction, what are the teacher behaviors or strategies that

will foster that goal? Researchers of teaching thus were not greatly concerned with defining desirable outcomes of schooling or considering fundamental questions about what students should be learning. In searching for measures of student outcomes, they worked from existing assumptions--prevalent in the research and schooling communities--about the nature of knowledge, relying for the most part on standardized tests of student achievement.

Teaching was viewed from this perspective as a delivery system for knowledge that was specified by others. Process-product researchers carried this implicit assumption with them even when they focused on the teaching of particular subject matters. Thus, when Good and Grouws (1979; and Good, Grouws, & Ebmeier, 1983) studied elementary school mathematics teaching, they continued in the process-product tradition by assuming that the best indicator of effective teaching is students' achievement test scores and by focusing on instructional behaviors that were virtually free of mathematics content. The resulting recommendations, such as "focus on meaning and promoting student understanding" or "assess student comprehension" might apply as well to the teaching of history or reading as to the teaching of mathematics. These rather generic recommendations suggest the assumption that teaching is a delivery system for content that is determined by others and specified in the curriculum.

Mathematics educators have criticized process-product researchers for their failure to consider what mathematics is worth knowing and for their ready acceptance of achievement tests as the primary measure of instructional effectiveness (Romberg & Carpenter, 1986). "Residualized mean gain scores have become methodological proxies for 'what we want,' and the standardized test has become the operational definition of what is worth knowing" (Romberg and Carpenter, 1986, p. 865). But what kind of operational definition of knowing mathematics do these tests provide and why were they so readily accepted by researchers on teaching?

Mathematics knowledge defined as achievement on standardized tests. In searching for objective measures of student learning, process-product researchers heartily embraced standardized achievement tests. These tests were designed to be relatively impervious to minor variations in curriculum and to measure learning outcomes valued by school systems and the public. Standardized tests provided measures of learning expressed as single numbers or small sets of numbers with relatively high reliability

that could be used as the outcome variables of process-product research. Having such a "clean" set of outcome measures was important for being able to describe the relationships among variables that were the focus of scientific research from this perspective. Researchers in the process-product tradition often expressed the caution that achievement tests measure only some of the important outcomes of schooling in limited ways, but that these tests were the best we have for providing relatively unbiased indicators of the effectiveness of instruction (e.g., Brophy & Good, 1986).

But many scholars have argued that standardized achievement tests represent a severely limited view of what mathematics is worth knowing (Romberg & Carpenter, 1986). There is too much emphasis on isolated computational skill. There is not enough problem solving, and what problem solving there is tends to consist of word problems that can be solved by simply applying learned algorithms. The tests do little or nothing to assess students' ability to comprehend mathematical reasoning in written communications, their ability to engage in mathematical argument, or their willingness to approach problems they encounter by drawing on quantitative tools--all outcomes that are emphasized in recent reform rhetoric (National Council of Teachers of Mathematics, 1989; National Research Council, 1989).

At the same time, standardized tests are not always so readily dismissed. Much of the heat generated over the sorry state of current mathematics education in the United States is fueled by pointing to students' dismal performance on standardized tests. Clearly, policy makers and the public are willing to accept test scores as an important indicator of what students are learning. But how do the assumptions about what it means to know mathematics and what mathematics is worth knowing that underlie these tests compare with those of the perspectives from psychology and mathematics that we have considered?

One point of mismatch is the tasks included in most tests. For the most part, tasks on standardized achievement tests involve isolated computation and the solving of routine word problems, resulting in a picture of knowing mathematics as knowing computational procedures and the skills needed to apply those procedures to a constrained set of problems. Our analysis of perspectives from the discipline of mathematics suggests a much broader range of tasks--tasks involving various kinds of nonroutine problem solving, the mathematization of situations and judgments about the appropriateness of mathematical models for various purposes, and the use of mathematical argument and justification.

Another point of mismatch is that the theories of testing upon which standardized achievement tests are constructed assume that knowledge can be regarded as a relatively stable trait or continuum along which individuals can be ordered (Anastasi, 1976; Cronbach, 1970). Test items are considered samples from relatively homogeneous pools of potential items representing what is to be known. These simplifying assumptions are important in providing the basis for powerful mathematical models underlying test construction, but they are not necessarily consistent with other views of knowledge. The assumptions are relatively compatible with traditional views of learning as the transmission of knowledge. From that perspective it makes sense to think of measuring how much knowledge a person has acquired. But the assumptions are not so readily consistent with cognitive views of learning and knowing that emphasize the role of knowledge structures actively constructed by the individual learner. What becomes important from this perspective is not how much knowledge a person has, but how that knowledge is organized and how accessible it is in various situations. This is not to say that testing is totally incompatible with cognitive views of learning, but that most existing tests were built on assumptions of knowledge as a more static entity.

A final assumption reflected in testing that may present a more fundamental dilemma is the assumption that knowledge can be decontextualized. To believe that a written test can provide a valid picture of an individual's knowledge, one must assume that the individual "carries" that knowledge within him or herself to the testing situation. The recent emphasis by some cognitive researchers on the situated nature of cognition raises the important question of whether it is, in principle, possible to assess a person's competence through tests that are, by their very nature, meant to measure decontextualized knowledge.

Knowledge as separable into discrete parts. Another assumption that is reflected not only in process-product research but in much of current practice in curriculum development and instruction is that knowledge can be decomposed into discrete entities. In curriculum, this assumption is reflected by the breaking up of content to be learned into sets of discrete objectives that are to be separately taught and tested. Some mathematics educators categorize knowledge into skills, concepts, and applications (Fey, 1982; Trafton, 1980). Many elementary mathematics textbooks treat two-digit subtraction with regrouping

as a skill distinct from two-digit subtraction without regrouping, and contain separate lessons on skills for estimating and problem solving. This splintering of the curriculum has been fueled by a variety of sources, including the general reductionistic assumptions of psychology, associationist, and behavioristic theories that consider knowledge to be collections of behaviors or bonds acquired essentially by accretion (Thorndike, 1922); instructional theories that emphasize the direct teaching of component skills that are then combined into more complex performances (Gagné et al., 1988); and the push to be as specific as possible in setting objectives for instruction (Bloom, Englehart, Furst, Hill, & Krathwohl, 1956; Tyler, 1949). This emphasis on separation of knowledge runs counter to the emphasis in cognitive views of learning and knowledge as structured and connected.

Researchers studying teaching have inherited assumptions about mathematics knowledge being decomposable and have built upon them in a number of ways. For example, researchers studying mathematics teaching have viewed problem solving as a matter of learning mathematical procedures and concepts, then learning the skills or strategies that are needed to apply the procedures and concepts to problems. Good et al. (1983) measured student learning with an achievement test that separated mathematics knowledge into three subtests: knowledge, skill, and problem solving. Good et al. expressed concern over the lack of difference between their treatment and control groups on the problem-solving subtest "because we felt that if mathematics knowledge is to be applied to "everyday" matters, students need skills in this area (e.g., to compare whether the 12 oz. or 16 oz. package is the better buy)" (p. 93, emphasis added). Similarly, Brophy (1988), in arguing that most of what is taught in school is amenable to the principles of active teaching, suggested that

> Students learning mathematics problem-solving skills need not only practice in applying procedural algorithms to well-structured problems but also modeling and explicit instruction in strategies for identifying relevant information and formulating poorly structured problems accurately, as well as strategies for analyzing, simplifying, and developing methods for solving unfamiliar problems. (p. 8)

These statements suggest the assumption that mathematical knowledge consists of fairly separable skills, concepts, and strategies that can be explicitly modeled and directly taught.

Students' engagement as a measure of learning. A final assumption of some researchers in the process-product tradition is that the time a student spends actively engaged in appropriate academic

tasks can serve as a proxy for the students' learning. This assumption has resulted in process-product researchers' focusing on a variety of time-related variables such as opportunity to learn, time-on-task, and academic learning time (Berliner, 1979; Rosenshine, 1979). Much of this emphasis on time spent in learning can be traced to Carroll's (1963) model of school learning, which included three variables for predicting school achievement that could be expressed in terms of time: aptitude (defined as the amount of time a student needs to learn a given task), opportunity to learn (time allowed for learning the task), perseverance (time student is willing to spend learning the task). As operationalized by classroom researchers, these time-related variables reflect the behaviorist sense of the word active in learning: A student is learning if he or she is overtly responding. Thus, if a student is visibly engaged in a task, he or she can be assumed to be learning that task.

Although measures of the time students spend engaged in academic tasks have proved successful as rough predictors of student achievement, many researchers from both inside and outside the process-product tradition have criticized this emphasis on time and observable engagement (e.g., Peterson, 1988; Romberg & Carpenter, 1986) with a number of arguments:

1. A focus on time spent and opportunity to learn emphasizes efficiency, quantity, and presence or absence when both what is learned (the mathematics) and the quality (of students' thinking) may be most important

2. Measures of time spent and opportunity to learn are behavioral measures that do not tap the cognitive processes and strategies in which the learner engages in the act of learning and which may define the essence of knowing, understanding, and problem solving in mathematics

3. Although empirical data exist to show a significant positive linear relationship between observed "engaged time" of the learner in low-level mathematics activities and tasks (knowledge, facts, and skill) and students' subsequent low-level mathematics achievement, empirical data do not show a linear positive relationship for "higher-level" mathematics activities, including mathematical applications and problem solving (see, for example, Swing, Stoiber, & Peterson, 1988)

4. Analyses based on "time spent" or on "content coverage" have led to a fragmented view of what is learned in mathematics--both the mathematical content and the aspects of mathematical knowledge, skill,

and problem solving (e.g., conceptual vs. procedural knowledge)--rather than leading to an integrated view that assumes connectedness among all these aspects of knowing mathematics.

Definition of mathematical knowledge in existing classroom practice. We have presented several assumptions underlying research on effective teaching that stand in contrast to assumptions about knowing and learning mathematics emerging from cognitive psychology and the discipline of mathematics. The assumptions of classroom researchers have been inextricably linked to existing classroom practice. This is due both to the influence of teaching effectiveness research on educational policy and practice, and to the derivation of effective teaching practices from existing classroom practice. The tight linkage of process-product research to existing classroom practice is both a strength and a weakness. As a strength, this grounding in classroom practice helps ensure that recommendations from the research will be practical: Because the recommended teaching behaviors were derived by observing teachers, it is reasonable to assume that other teachers can carry out the behaviors as well. In addition, these studies of teaching and learning are situated, not in laboratory changes. Thus, Romberg and Carpenter (1986) contended [redacted]ariables of process-product settings, but in the classroom contexts researchers hope to apply them to.

But in terms of weaknesses, process-product researchers have been criticized for being inherently conservative in seeking to improve existing instructional practices, rather than to seek more fundamental changes. Thus, Romberg and Carpenter (1986) contended that the variables of process-product research "can only make current teaching more efficient or effective, but they cannot make it radically different" (p. 865). Researchers on teaching counter with one of their original motivating concerns: concern about attempts to make mathematics teaching radically different without empirical data based on observations of classroom practice [redacted]). They point out the importance of seeing whether major changes "will work" in the classroom before calling for massive changes in practice.

As researchers have responded to the criticisms of process-product research by building new research programs to study teaching (Shulman, 1986), they have, for the most part, maintained their commitment to studying existing classroom practice. This commitment is motivated in part by the recognition that there are considerable improvements that can be made within the pervasive frameworks

of teaching in existing practice, in part by understanding that more radical changes in teaching must start with a rich understanding of existing practice if they are to be successfully realized. For example, Leinhardt (1988) has used constructs and methods from cognitive science to study the teaching and learning of mathematics by expert teachers, selecting the teachers primarily on the basis of consistent gains in student achievement. Leinhardt's analyses have revealed important teacher knowledge of mathematics subject matter and of the structures and routines for conducting good lessons, helping us to better understand expert "traditional" teaching. Although Leinhardt (1988) argued for the need to study "dramatically different teaching styles and lesson patterns," she reaffirmed the traditional criterion and kind of student learning used by researchers on teaching effectiveness as teachers who "enrich the students' concepts, concrete experiences, and extended problem-solving capabilities while not abandoning the computational aspects of arithmetic education that society seems to value" (p. 65).

Broadening Conceptions of Mathematical Knowing in Classrooms

The views of teaching and learning that seem to underlie much of current mathematics instruction in elementary school classrooms entail "teaching as telling," and learning as received knowing (Belenky, Clinchy, Goldberger, & Tarule, 1986), "where teachers supply information and show how to perform procedures, and students accept this knowledge, rather than arriving at it through their own constructive intellectual and social activity" (Greeno, 1989, p. 137). Currently, a number of researchers are working in elementary school classrooms to explore instruction based on alternatives to these fundamental assumptions about teaching and learning (Carpenter et al., 1987; Cobb, Wood, & Yackel, in press; Lampert, 1988a). They are trying in various ways to bring alternative assumptions about knowing and learning mathematics from psychology and the discipline of mathematics into the classroom, while respecting the many constraints of classroom teaching.

Cognitively guided instruction. Carpenter, et al. (1987; see also Carpenter, Fennema, Peterson, Chiang, & Loef, 1988) have sought to help first-grade teachers change their underlying views of learning in light of cognitive research on individual children's solving of addition and subtraction word

problems. Their instructional framework, which they call cognitively guided instruction (CGI) is based on two major assumptions derived from cognitive studies of learning:

> One is that instruction should develop understanding by stressing relationships between skills and problem solving with problem solving serving as the organizing focus of instruction. The second assumption is that instruction should build on students' existing knowledge. (Carpenter et al., 1988, p. 11)

These assumptions parallel two of the themes from cognitive psychology that we considered above: connections among types of knowledge and learning as active construction of knowledge. Thus, the CGI researchers' assumptions about learning and knowing mathematics are clearly grounded in psychological research on individual learners. In helping teachers apply these perspectives about individual learning to their teaching, the CGI researchers draw on research on teacher thinking (Clark & Peterson, 1986), which assumes that

> Classroom instruction is mediated by teachers' thinking and decisions. Thus, researchers and educators can bring about the most significant changes in classroom practice by helping teachers to make informed decisions rather than by attempting to train them to perform in a specified way. (Carpenter et al., 1988, p. 10)

Thus the route to changing instructional practice is to change teachers' knowledge, beliefs, and attitudes. In particular, Carpenter et al. (1988) sought to provide teachers with knowledge about types of addition and subtraction problems (see Table 1) and the strategies that students typically use to solve these problems as they pass through general levels of expertise. They also urged teachers to incorporate broad instructional principles that emphasized making instruction appropriate for each student by basing instructional decisions on frequent assessment of the student's solution strategies.

Carpenter et al. (1988) found that teachers were able to learn about the research-based addition and subtraction problem types and solutions, and to use this knowledge in their instruction. CGI teachers were more likely to attend to their students' solution strategies than were control-group teachers. Students in CGI classrooms spent more time solving problems and were more successful in solving the sorts of complex addition and subtraction problems that have been the focus of the psychological research in this domain.

Constructivist teaching of second-grade mathematics. In another attempt to alter teachers' beliefs and instructional practices to reflect a particular perspective on learning, Cobb and his

colleagues (Cobb et al., in press; Cobb, Yackel, & Wood, 1988) have worked with second-grade teachers to alter their teaching to be consistent with a constructivist view of learning. Cobb and his colleagues began with the fundamental assumption that mathematics learning occurs as "individuals each construct their individual mathematical worlds by reorganizing their experiences in an attempt to resolve their problems" (Cobb et al., 1988, p. 93). The dilemma faced in applying this constructivist view of learning to teaching is resolving the tension between this position and the need to meet "institutionally sanctioned goals of instruction" (p. 94)--for students to learn particular, accepted mathematics. To temper without entirely resolving this dilemma, Cobb and his colleagues took a second, complementary, perspective from anthropology: that mathematical knowledge is cultural knowledge, which is best fostered by "nurturing a classroom atmosphere that encourages the negotiation of meaning" (p. 102).

Cobb and his colleagues entered the classroom by way of instructional activities and materials that would facilitate teaching that was compatible with a constructivist perspective by providing rich opportunities for students to construct their own mathematical knowledge. The instructional materials were based on constructivist researchers' models of the construction of arithmetical knowledge by individual children (Steffe & Cobb, 1988; Steffe et al., 1983) and were designed to (a) make sense simultaneously to children at a variety of levels, (b) avoid an arbitrary separation of conceptual and procedural knowledge, (c) address traditional second-grade learning objectives (i.e., those measured on standardized achievement tests), and (d) facilitate sustained whole-class discussions about mathematics (Cobb et al., 1988). The materials were accompanied by a strong commitment to treat teachers, as well as students, as constructivist learners and, thus, an avoidance of instructional prescriptions. Cobb and his colleagues worked closely with a second-grade teacher using the materials to develop classroom management and interaction routines that would facilitate meaningful interaction structured around the instructional materials. Thus, like Carpenter et al. (1987), Cobb et al. began with a particular view of what it means to know and learn mathematics from the psychological perspective of the individual learner and are working with teachers to develop forms of instruction that are compatible with that view.

Bringing mathematical discourse and argument to a fifth-grade classroom. In the classroom Lampert (1988a) is also exploring alternatives to pervasive views of knowing and learning

mathematics, "in which doing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher" (p. 135). But rather than taking as her primary starting point a particular view of learning from psychological research on the individual learner, Lampert begins with assumptions about the nature of knowledge and knowledge growth within the discipline of mathematics. She argues, following Kramer (1970), that, "in mathematics, authority comes from agreeing on shared assumptions and reasoning about their consequences" (Lampert, 1988a, p. 135). Thus, according to this view, the discourse in classrooms should be more like the discourse of argument and conjecturing that takes place within the discipline of mathematics, with a shift in authority for what constitutes valid mathematical knowledge from teacher decree to the sense-making and reasoning of the individual. "Lessons will be in the form of a mathematical argument, which students accept or reject on the basis of their own reasoning" (p. 136).

As a teacher-scholar studying her own teaching of fifth-grade mathematics, Lampert (1988a) is developing a classroom pedagogy in which an important goal is for teachers and students to engage in mathematical argument and discourse. Her research in this setting focuses on what is involved from the teacher's perspective in teaching mathematics with this goal and what "students' understanding look[s] like in the social context of the public school" (p. 132). Thus, although informed by psychological research, Lampert does not begin with psychological assumptions about the learning of individuals and attempt to apply them to the classroom context; rather, she uses perspectives from psychology as one lens to consider the nature of students' understanding and knowing of mathematics in classroom contexts.

Carpenter et al. (1988), Cobb et al. (1988), and Lampert (1988a) are all attempting to change rather fundamental assumptions about teaching and learning that are evidenced in most classrooms. Carpenter et al. (1988) and Cobb et al. (1988) approach this challenge by working with teachers to develop instructional strategies that are compatible with particular views of learning emerging from psychological studies of individuals learning mathematics. Carpenter et al.'s perspective draws heavily on a specific body of research--that of children's solving of addition and subtraction word problems--and emphasizes the principles from cognitive psychology of learning as an active process that builds on existing

knowledge and of the importance of emphasizing connections and relationships among ideas. For Cobb et al., the key is developing instruction that is compatible with the constructivist assumption that children construct their own mathematical knowledge through interaction in social contexts. Lampert (1988a), in contrast begins with perspectives from mathematics and classroom teaching to develop a pedagogy in which teachers and students engage in mathematical argument rather than establishing mathematical truth by decree.

Summary and Conclusions: Goals for Mathematics Learning in Elementary Schools

We began this paper with two beliefs about why it is important to learn mathematics: the role of mathematical tools and ways of thinking in todays society and workplace and appreciation of mathematics as a great cultural achievement. How can the diverse perspectives we have considered inform our thinking about what mathematics is worth knowing and how it might be learned in elementary schools? Summarizing or even synthesizing the diverse perspectives we have considered--from the psychology of the individual learner, from the discipline of mathematics, and from classroom practice--will not provide ready answers to these questions. This is due, in part, to these different perspectives representing some fundamental issues about the nature of mathematical knowledge and learning, such as the tension between mathematical knowledge as a veridical description of an external reality versus mathematical knowledge as a human cultural construction, and the tension between socially accepted definitions of mathematical knowledge and the need for individual learners to construct their own mathematical meanings. So, rather than providing answers, our hope is that our consideration of these multiple perspectives may clarify our questions and turn our attention to issues that might otherwise be overlooked.

In thinking about what students should be learning about mathematics in elementary schools and how that learning might take place, one could begin with the notion of mathematics as powerful set of tools evolved over time in the culture. This notion is supported by our discussion of representation systems that can be said to exist in the public arena and serve as tools for communicating with others and that are also acquired by individuals where representations serve as tools for thinking. If we ignore perspectives

from psychology, we might suggest that decisions about what to teach in elementary school become decisions about what mathematical tools are important for students to learn. These tools can then be presented by teachers to be acquired by students. But there are least two difficulties with this approach. First, important aspects of what is important to learn about tools are lost when they are presented; that is, the largely implicit knowledge about when and how these tools are used for various purposes. Cognitive psychology has demonstrated that learners will filter the knowledge they "receive" through their existing, possibly idiosyncratic, knowledge structures--that they will impose their own meaning on what they are learning based on their existing conceptions and misconceptions. Thus one cannot assume that what a teacher presents is what a student will learn (Norman, 1980).

This view of the learner as actively imposing meaning and constructing his or her own knowledge can easily lead to the position that constructing mathematical knowledge is a highly personal affair and that the criteria for what constitutes legitimate or useful mathematical knowledge is that which has been constructed from existing knowledge of the individual. But what is to prevent every individual from constructing his or her own idiosyncratic mathematics that does not facilitate communication with others? How are individuals going to construct the shared mathematical tools that "it took humanity thousands of years to construct?" (Sinclair, 1988, p. 1). This is a fundamental tension that must be addressed in any successful approach to schooling in mathematics. Concentrating solely on the personal meanings of mathematics that learners construct and building on the knowledge they bring to the learning setting creates the risk of ignoring the importance of mathematical conventions, conventions that must be learned as they are used in our society if they are to serve as powerful tools for mathematical thinking and communication. Concentrating too heavily on the mathematics to be learned--on presenting conventional tools--while ignoring the role played by the individual's existing conceptions and efforts at sensemaking of what is learned, creates the risk that the tools will not be learned at all or that they will not be learned in ways that make them accessible to the learner when needed in various situations.

How can school practice be shaped to deal with this tension and others that are created in attending to the diverse perspectives that arise from attending to psychology, mathematics, and the classroom context? We propose that there are three aspects of knowing mathematics that must be attended to in

forging a school practice successful in fostering powerful mathematical understanding. At one level these aspects can be thought of as the desired outcomes of a successful educational program--as the goals of instruction. But characterizing these aspects of knowing mathematics as outcomes suggests an artificial separation of outcomes from the educational process. Rather than separate process from outcomes, we present these themes as desired characteristics of the practices of teaching and learning in school. Since such practices have rarely been tried, and even more rarely studied by researchers, we have much to learn about how they might affect the outcomes of schooling.

If teaching and learning in school are to lead to genuine academic accomplishments in mathematics, it is reasonable to assert that they must be built around the corpus of "big ideas" that are fundamental to mathematics and mathematical thinking--ideas like place-value, part-whole relationships, the notion that numbers related as a function will make a line on a graph. These represent knowledge that is difficult to specify explicitly and precisely and that must be constructed by the individual. These ideas and the relationships among them constitute what Vergnaud has referred to as conceptual fields. Although teachers can present various descriptions of these ideas to students, they cannot "tell" these ideas in all their complexity to students; rather they can create classroom activities which provide students with opportunities to construct them. The construction or development of these ideas is an ongoing process over the course of elementary schooling; the ideas are revisited and continually refined in a variety of settings. At the same time that we recognize these ideas are made meaningful through individual construction, however, we also take note of the fact that the ideas are available to us because they have been invented and recognized as important over the long history of mathematics as a discipline.

In addition to these conceptual big ideas there are conventions of mathematics that must essentially be presented to students, including many of the symbols, representations, and other tools of mathematics, for example the labels "one", "two", "three". . . for digits, knowing that the symbol "%" means percent, or knowing that the x and y axes are typically used in a graph. It does not make sense to think of students "constructing" knowledge of such conventions--they have to be learned and remembered as they are conventionally used or they do not play their important role in facilitating mathematical communication. Research in cognitive psychology suggests that such things will be

remembered better if they are made meaningful to the learner in the sense of relating to the learner's existing knowledge structures, but they need to be learned pretty much as is. The line between these conventions of mathematics that must be acquired and the big ideas that must be constructed is by no means a distinct one. Such pervasive concepts as place-value are intricately connected to the conventions of the representation systems we use. The convention is a tool which can be used to learn and think about the concept. If the conventional representation systems with which we deal with quantity did not involve place value, place value would not constitute a fundamental mathematical concept for students to build during their elementary school years.

Besides big ideas and conventions, a third important aspect of mathematical practice in schools needs to be personal sensemaking. It is important for students to have the disposition of continually trying to figure things out and make sense of them. It is important for elementary classrooms to be places where thinking about ideas and making personal sense of them is valued. The importance of sensemaking is supported by both psychology and mathematics. From the perspective of cognitive psychology, students will not construct important mathematical understandings or acquire important conventions unless they actively work to integrate new information and experiences with their existing knowledge. In the discipline of mathematics, the fundamental warrant for determining what is valid mathematical knowledge is not empirical evidence or decree by authority, but whether that knowledge can be shown to derive logically from agreed-upon assumptions. It is important for classrooms to provide settings in which students' attempts to make sense of new ideas are valued and explored and their current ways of thinking are valued and examined not ignored, and where mathematical conclusions are supported by reasoned argument rather than teachers or answer books.

In shaping classroom practice to attend to these three aspects of knowing mathematics, there are some features of instruction that are strongly supported by all three of the general perspectives we have discussed--psychology, mathematics, and classroom practice. First is the importance of talking about mathematics. In contrast to current classroom practice in which much of the activity involves students practicing procedures that have been explained or modeled by the teaching, classrooms should provide ample opportunities to verbalize their thinking and to have conversations about mathematical ideas and

procedures. If we view understanding of mathematics from the perspective of cognitive psychology as the knowledge structures of the individual, but we view school as the place where individuals acquire publicly valued knowledge, then individual knowledge must be refined and revised in the social setting of the classroom. Students must learn to communicate their ideas to one another and their teachers must learn to communicate with students in terms of mathematical arguments. Verbal discourse, as our primary mode of communication, constitutes an important means of revealing individual knowledge. If we view knowledge and thinking as inherently situated in social and physical contexts, much of what is learned about mathematics is implicit; we therefore need to make communication about and around mathematical activity as rich as possible to serve as the site for learning through participation in that activity. From the discipline of mathematics, we have seen the importance of conjecturing and defending ideas in mathematics--activities which require extended public discourse. If the teacher adopts mathematical conventions about how to justify and refine assertions, he or she will need to engage in discussions with students, challenging these with counterexamples rather than with judgments about the wrongness of their answers.

A second aspect of practice that follows from our analysis is the importance of considering the kinds of mathematical activity occurring in classrooms. From psychology comes the notion that the kinds of activity in which learners engage is critically important in the knowledge structures they construct or acquire. Our discussion of the discipline of mathematics revealed that the traditional curriculum and much of the research in psychology has focused on rather limited aspects of mathematical activity. But what kind of mathematical activity is appropriate for elementary-school classrooms? The goal of elementary school mathematics education is not for all students to become professional mathematicians, just as we do not teaching reading and writing in hopes that all students will become novelists or editors, or social studies so that all students will become historians. But there is a sense in which we want all children to come to appreciate what it means to think like a mathematician. Mathematics offers powerful ways of thinking about the world and these ways of thinking increasingly pervade our culture. It is essential that all people be able to communicate and reason about quantitative relationships as part of being a literate, participating member of our information-oriented society.

As put forth in Everybody Counts, "elementary school is where children learn the mathematical skills needed for daily life" (National Research Council, 1989, p. 46). But the mathematical skills needed for daily life have changed dramatically in recent years. It is no longer enough to be proficient at the isolated arithmetic calculation that pervades current curriculum and instructional practice. Full participation in today's information-oriented society requires much more than the computational skill that has been the soul and substance of traditional elementary school mathematics. In addition to having all students appreciate what kind of knowledge they have when they know mathematics, we want to give students enough of a feeling for what the discipline is like to enable them to make informed choices about whether to pursue it in a more concentrated way. Since our education system is structured so that students may choose to virtually "opt out" of mathematics at the secondary level, these issues must be addressed in elementary curriculum and instruction.

Finally, thinking about elementary classrooms as places where rich discourse about mathematics takes place, and where students can participate in the kind of mathematical activity similar to that engaged in by mathematically literate citizens, raises the crucial issue of what teachers in such classrooms need to know. First, it becomes apparent that making the classroom a place where students can work with and acquire the powerful cognitive tools of mathematics requires that the teacher must be one who uses these tools him or herself. As our analysis of knowing in the discipline has shown, mathematics in use and the canons of mathematical argument could never be completely specified in the form of a pre-packaged curriculum that would permit conveying them in their entirety to students.

> Teachers themselves need experience in doing mathematics--in exploring, guessing, testing, estimating arguing, and proving--in order to develop confidence that they can respond constructively to unexpected conjectures that emerge as students follow their own paths in approaching mathematical problems. Too often, mathematics teachers are afraid that someone will ask a question that they cannot answer. Insecurity breeds rigidity, the antithesis of mathematical power. (National Research Council, 1989, p. 65)

Teachers' need not only rich knowledge of mathematics conventions and "big ideas," but also the personal disposition of sense-making. But the teacher's own knowledge and beliefs about mathematics cannot simply be "told"--the teacher also needs to be attentive to students' ways of knowing and understanding mathematics so that their understandings can be shaped and guided.

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